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Microlocal study of topological Radon transforms and real projective duality

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Abstract

Various topological properties of projective duality between real projective varieties and their duals are obtained by making use of the microlocal theory of (subanalytically) constructible sheaves developed by Kashiwara [M. Kashiwara, Index theorem for constructible sheaves, Astérisque 130 (1985) 193–209] and Kashiwara–Schapira [M. Kashiwara, P. Schapira, Sheaves on Manifolds, Grundlehren Math. Wiss., vol. 292, Springer, Berlin–Heidelberg–New York, 1990]. In particular, we prove in the real setting some results similar to the ones proved by Ernström in the complex case [L. Ernström, Topological Radon transforms and the local Euler obstruction, Duke Math. J. 76 (1994) 1–21]. For this purpose, we describe the characteristic cycles of topological Radon transforms of constructible functions in terms of curvatures of strata in real projective spaces.

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1. Introduction

Since the beginning of the theory, projective duality has been one of the main themes in algebraic geometry. Many mathematicians were interested in the mysterious relations between the topologies of complex projective varieties V and their duals V^*. Above all, they observed that the tangency of a hyperplane $H \in V^*$ with V is related to the singularity of the dual V^* at H. For example, an inflection point of a plane curve C corresponds to a cusp of the dual curve C^*, and a bitangent (double tangent) $l \in C^*$ of C is an ordinary double point of C^*. The most general results for complex plane curves were found in the 19th century by Klein, Plücker and Clebsch, etc. (see, for example, [30, Theorem 1.6] and [34, Chapter 7], etc.). This beautiful symmetry between complex plane curves and their duals was extended to higher-dimensional cases in the last two decades. In particular, after the important contributions by Viro [33], Dimca [7] and Parusinski [25], etc., in 1994 Ernström [8] proved the following remarkable result. For an algebraic variety V, denote by $\mathrm{Eu}_V : V \to \mathbb{Z}$ the Euler obstruction of V introduced by Kashiwara [15] and MacPherson [20] independently. Recall that this constructible function Eu_V measures the singularity of V at each point of V and takes the value $1 \in \mathbb{Z}$ on the regular part of V. If we take a Whitney stratification $\bigsqcup_{\alpha \in A} V_\alpha$ of V, then the value of this function on a stratum $V_{\alpha'}$ is defined by those on V_α's such that $V_{\alpha'} \subset \overline{V_\alpha}$ (see [16] for the detail).

Theorem 1.1. [8, Corollary 3.9] *Let V be a smooth projective variety in a complex projective space of dimension n and V^* its dual. Take a generic hyperplane H, i.e. a hyperplane not belonging to V^*. Then for any hyperplane $H' \in V^*$, we have*

$$\chi(V \cap H') - \chi(V \cap H) = (-1)^{n-1+\dim V - \dim V^*} \mathrm{Eu}_{V^*}(H').$$

Namely the jumping number $\chi(V \cap H') - \chi(V \cap H)$ of the Euler–Poincaré index of the hyperplane section $V \cap H'$ by $H' \in V^*$ is expressed by the singularity $\mathrm{Eu}_{V^*}(H')$ of the dual variety V^* at $H' \in V^*$. Moreover we have now several important theorems (the class formulas) which express the degree of the dual V^* in terms of the topological data of V. For these results, we refer to an excellent review in Tevelev [30, Chapter 10] (see also [22]).

The aim of this paper is to initiate the microlocal study of projective duality for real projective varieties, and prove analogous results in the real analytic setting. Let $X = \mathbb{P}_n$ be a real projective space of dimension n and $Y = \mathbb{P}_n^*$ its dual. Recall that Y is naturally identified with the set of hyperplanes H in X. In this situation, for a smooth real analytic submanifold M of X we shall define its dual M^* as a closed subanalytic subset of Y in the following way. Let $S = \{(x, H) \in X \times Y \mid x \in H\}$ be the incidence submanifold of $X \times Y$ and consider the maps $f : S \to X$, $g : S \to Y$ induced by the projections from $X \times Y$ to X and Y respectively. We use the notation $\dot{T}^*X$ to denote the cotangent bundle T^*X with its zero section $T^*_X X$ removed. Similarly we denote $\dot{T}^*Y$, $\dot{T}^*_S(X \times Y)$, etc. Consider the diagram

$$\begin{tikzcd} & T^*_S(X \times Y) \arrow[dl, "\overline{p_1}"'] \arrow[dr, "\overline{p_2}^a"] & \\ T^*X & & T^*Y \end{tikzcd}$$

in which $\overline{p_1}$ and $\overline{p_2}^a$ are induced by the maps

$$p_1 : T^*X \times T^*Y \to T^*X,$$
$$p_2^a : T^*X \times T^*Y \underset{p_2}{\longrightarrow} T^*Y \underset{\alpha_{T^*Y}}{\overset{\sim}{\longrightarrow}} T^*Y$$

(α_{T^*Y} is the antipodal map of T^*Y). Then outside the zero sections we obtain isomorphisms

$$\begin{tikzcd} & \dot{T}_S^*(X \times Y) \arrow[dl, "\widetilde{p_1}"', "\sim"] \arrow[dr, "\widetilde{p_2}^a", "\sim"'] & \\ \dot{T}^*X & & \dot{T}^*Y. \end{tikzcd}$$

We set $\Phi = \widetilde{p_2}^a \circ \widetilde{p_1}^{-1} : \dot{T}^*X \overset{\sim}{\longrightarrow} \dot{T}^*Y$. In the standard affine charts of X and Y, the isomorphism Φ thus obtained is nothing but the classical Legendre transform. Let us now define the dual $M^* \subset Y$ of the smooth real analytic submanifold $M \subset X$ by $M^* = (\dot{\pi}_Y \circ \Phi)(\dot{T}_M^* X)$, where $\dot{\pi}_Y : \dot{T}^*Y \to Y$ is the projection. Then M^* is a closed subanalytic subset of Y. Namely, our dual M^* is a projection of a smooth conic Lagrangian submanifold $\Lambda = \Phi(\dot{T}_M^* X)$ of $\dot{T}^*Y$ to the base space Y. The study of singular sets defined in this way, called Legendre singularities, has a long history. They were precisely studied especially in relation with the theory of geometric optics (see for example Arnold [1], Arnold, Gusein-Zade and Varchenko [2] and Urabe [32], etc.). Since we are working in the subanalytic category, the dual $M^* = \dot{\pi}_Y(\Lambda)$ has moreover the following desirable property. Denote by M^*_{reg} the regular (or smooth) part of M^*. Then we have $\Lambda = \overline{T^*_{M^*_{\mathrm{reg}}} Y}$ in $\dot{T}^*Y$. This result will be proved in Section 5 with the aid of μ-stratifications introduced in [19, Definition 8.3.19] by Kashiwara–Schapira. Note that our definition of duals coincides with the usual (classical) one. Namely, our dual M^* is the trajectory of hyperplanes $H \in Y = \mathbb{P}_n^*$ in $X = \mathbb{P}_n$ tangent to M.

Now let us define a (subanalytically) constructible function $\varphi_M : Y \to \mathbb{Z}$ on Y by assigning to each hyperplane $H \in Y$ the Euler–Poincaré index $\varphi_M(H) = \chi(M \cap H)$ of the hyperplane section $M \cap H$. Our objective is to study this important function in the light of the microlocal theory of $\mathbb{R}$-constructible sheaves (functions) invented by Kashiwara [17] and Kashiwara–Schapira [19]. More precisely, if we denote by $CF(X)$ (respectively $CF(Y)$) the abelian group of (subanalytically) constructible functions on X (respectively on Y) and consider the topological Radon transform

$$\mathcal{R}_S : CF(X) \to CF(Y)$$

defined by $\mathcal{R}_S(\varphi) = \int_f g^*\varphi$ for $\varphi \in CF(X)$ (see [19, Chapter IX] or Sections 2 and 3 for the definitions of $\int_f$ and g^*), then the function φ_M is equal to the topological Radon transform $\mathcal{R}_S(\mathbf{1}_M)$ of the characteristic function $\mathbf{1}_M \in CF(X)$ of M. Thus, the full machinery of the microlocal theory of sheaves can be applied to study the function $\varphi_M = \mathcal{R}_S(\mathbf{1}_M)$. In Theorems 5.13 and 5.14, we show that if the value $\varphi_M(H) = \chi(M \cap H)$ of the function φ_M at a point $H \in Y \setminus M^*$ is given then we can express the value $\varphi_M(H') = \chi(M \cap H')$ at any $H' \in M^*$ in terms of the singularity of the dual set M^* at H'. In order to state a part of this main result quickly, assume moreover that the values of φ_M on $Y \setminus M^*$ and M^*_{reg} are already given. Then we can completely determine the whole function φ_M by the following theorem. For a relatively compact subanalytic open subset U in Y, we define the Euler integral $\int_U \varphi$ of $\varphi \in CF(Y)$ over U by $\int_U \varphi = \chi(U; F)$, where $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ is an $\mathbb{R}$-constructible sheaf such that the function obtained by taking the local Euler–Poincaré indices of F is φ. In fact, we can define this integral without using sheaves. However for this purpose, we have to slightly generalize the definition of the usual Euler integrals of [19, Chapter IX]. See Definition 5.10 for the detail.

Theorem 1.2. *Let $Y = \bigsqcup_{\alpha \in A} Y_\alpha$ be a μ-stratification* (*for the definition see* [19, Definition 8.3.19]) *of Y consisting of connected strata Y_α's and adapted to M^*. Then the function φ_M is constant on each stratum Y_α. Assume moreover that for $k \geqslant \operatorname{codim} M^*$, the values of φ_M on the strata Y_α's such that $\operatorname{codim} Y_\alpha \leqslant k$ are already determined. Then for any stratum Y_β such that $\operatorname{codim} Y_\beta = k+1$, the value of φ_M on Y_β is given by*

$$\int_{B(y,\varepsilon)\cap\{\psi<0\}} \varphi_M, \tag{1.1}$$

where $B(y,\varepsilon)$ is a small open ball centered at a point $y \in Y_\beta$ and ψ is a real-valued real analytic function defined in a neighborhood of y satisfying $\psi^{-1}(0) \supset Y_\beta$ and

$$\bigl(y; \operatorname{grad}\psi(y)\bigr) \in \dot{T}^*_{Y_\beta} Y \Bigl\backslash \Bigl(\bigcup_{\alpha \neq \beta} \overline{T^*_{Y_\alpha} Y} \Bigr).$$

Note that, thanks to [19, Corollary 8.3.24] we can always take a function ψ satisfying the above conditions. Moreover, since by the result [31] of Trotman the μ-condition of [19] is equivalent to the so-called w-regularity condition, we can easily calculate the Euler integral (1.1) above by using a constructible sheaf over the open semi-ball $B := B(y,\varepsilon) \cap \{\psi < 0\}$ which corresponds to $\varphi_M|_B$. Namely, with the topological Euler characteristics χ, the Euler integral (1.1) is expressed by

$$\sum_{\alpha:\ Y_\alpha \cap B \neq \emptyset} \varphi_M(Y_\alpha)\bigl\{\chi(\overline{Y_\alpha} \cap B) - \chi(\partial Y_\alpha \cap B)\bigr\}.$$

Comparing the above recursive formula for the determination of $\varphi_M = \mathcal{R}_S(\mathbf{1}_M)$ with the definition of Euler obstructions in [16], we find a striking similarity between this theorem and Ernström's one. Indeed our theorem is proved along the same line as in [8], although the proof requires more general theories developed by [17,19]. First, to the constructible function $\mathbf{1}_M \in CF(X)$ (respectively $\varphi_M = \mathcal{R}_S(\mathbf{1}_M) \in CF(Y)$), we associate a conic subanalytic Lagrangian cycle, called the characteristic cycle of $\mathbf{1}_M$ (respectively of φ_M), in the cotangent bundle T^*X (respectively T^*Y). Then we see that the characteristic cycle of $\mathbf{1}_M$, i.e. the conormal cycle $[\dot{T}^*_M X]$ is sent to a multiple $\varepsilon[\Phi(\dot{T}^*_M X)]$ ($\varepsilon = \pm 1$) of $[\Phi(\dot{T}^*_M X)] = [\overline{\dot{T}^*_{M^*_{\mathrm{reg}}} Y}]$ by the microlocal Radon transform $\mathcal{R}^\mu_S = (g_* \circ f^*)$ (for the definition, see Sections 2, 3 or [19, Chapter IX]). In fact, this is just a very special case of the following more general result. Denote by $\mathscr{L}_X$ and $\mathscr{L}_Y$ the sheaf of conic subanalytic Lagrangian cycles in T^*X and T^*Y respectively. Then it is well known that the microlocal Radon transform $\mathcal{R}^\mu_S$ induces an isomorphism

$$\Gamma(\dot{T}^*X; \mathscr{L}_X) \xrightarrow{\sim} \Gamma(\dot{T}^*Y; \mathscr{L}_Y)$$

(see Proposition 3.4). In Section 4, combining the methods of the second fundamental forms developed by Griffiths–Harris [11] and Fischer–Piontkowski [9], etc. with the theory of simple sheaves of Kashiwara–Schapira [19], we describe the sign change $\varepsilon = \pm 1$ above in terms of the principal curvatures of M with respect to the canonical metric of the real projective space $X = \mathbb{P}_n$ (a similar calculation for topological integral transforms associated with the Legendre transform was done also in [29]). Also this result will be proved in a more general setting. Namely consider a general constructible function φ on X and take a μ-stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X adapted to φ. Then, in Theorem 4.5, we completely describe the signs in the characteristic cycle of $\mathcal{R}_S(\varphi)$ by the principal curvatures of the strata X_α's. It would be clear that these results are the "real" analogues of previous ones obtained by Brylinski [4], D'Agnolo–Schapira [6] and Ernström [8], etc. in the complex case. Finally, to write down the constructible function $\varphi_M = \mathcal{R}_S(\mathbf{1}_M)$ by its characteristic cycle $\mathcal{R}_S^\mu([\dot{T}_M^* X]) = \varepsilon[\overline{\dot{T}^*_{M^*_{\mathrm{reg}}} Y}]$ and prove Theorem 1.2, our slight modification of Euler integrals in Definition 5.10 will be effectively used. If we apply the same arguments to the complex case, we can give a more transparent new proof to the main theorems of Ernström [8] (see Remark 4.6). Note that in the real case the codimension of the dual set M^* in $Y = \mathbb{P}_n^*$ is usually equal to one. So, in general the values of φ_M on various connected components of $Y \setminus M^*$ may be different from each other. This means that the real projective duality treated in this paper is much subtler than the complex case studied by [8]. However in Theorem 5.14, we completely describe these differences in terms of the curvatures of M^* in $Y = \mathbb{P}_n^*$.

Finally, let us mention that in our forthcoming papers, the results obtained in this paper will be extended into various directions. In [24], we study topological Radon transforms defined by Euler integrals over lower-dimensional linear subspaces in $X = \mathbb{P}_n$. Also some generalizations of class formulas for complex projective varieties were proved in [22,23].

2. Review and preliminaries

In this section, we briefly recall the theory of characteristic cycles of constructible sheaves and constructible functions. The main reference is [19, Chapter IX] by Kashiwara–Schapira and we will follow the terminology in it throughout this paper.

For example, for a topological space X, we denote by $\mathbf{D}^b(X)$ the derived category of bounded complexes of sheaves of $\mathbb{C}_X$-modules. We also denote by $\omega_X \in \mathbf{D}^b(X)$ the dualizing complex of X defined by $\omega_X := a_X^!(\mathbb{C}_{\{\mathrm{pt}\}})$, where $a_X : X \to \{\mathrm{pt}\}$ is the map to a point. If X is a topological manifold of dimension n, this complex of sheaves on X satisfies the condition

$$H^j(\omega_X) \simeq \begin{cases} 0, & j \neq -n, \\ or_X, & j = -n, \end{cases}$$

where or_X is the orientation sheaf of X. Namely we have an isomorphism $\omega_X \simeq or_X[n]$ in $\mathbf{D}^b(X)$ in this case.

2.1. Characteristic cycles

Let X be a real analytic manifold and $\mathscr{F}$ a sheaf of $\mathbb{C}_X$-modules on it. Then we say that $\mathscr{F}$ is an $\mathbb{R}$-constructible sheaf if there exists a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X by smooth subanalytic manifolds X_α's such that the restriction $\mathscr{F}|_{X_\alpha}$ to each stratum X_α is a locally constant sheaf of finite rank over $\mathbb{C}_{X_\alpha}$. In this paper, whenever we consider a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X, we assume that any stratum X_α in it is connected. We denote by $\mathbf{D}^b_{\mathbb{R}-c}(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of objects with $\mathbb{R}$-constructible cohomology sheaves.

In 1985, Kashiwara [17] proved a beautiful formula which expresses the global Euler–Poincaré index

$$\chi(X;F)=\sum_{j\in\mathbb{Z}}(-1)^j \dim H^j(X;F)$$

of $F\in \mathbf{D}^b_{\mathbb{R}-c}(X)$ in terms of a Lagrangian cycle $CC(F)$ in the cotangent bundle T^*X of X. More precisely, this Lagrangian cycle is an element of the local cohomology group $H^0_\Lambda(T^*X;\pi_X^{-1}\omega_X)$ supported by a closed conic (i.e. $\mathbb{R}_{>0}$-invariant) subanalytic Lagrangian subset $\Lambda\subset T^*X$, where $\pi_X: T^*X\to X$ is the projection. Since we have

$$H^0_\Lambda\big(T^*X;\pi_X^{-1}\omega_X\big)\simeq H^n_\Lambda\big(T^*X;\pi_X^{-1}or_X\big)\quad (n=\dim X=\operatorname{codim}_{T^*X}\Lambda),$$

$CC(F)$ is locally a top-dimensional Borel–Moore homology cycle in Λ. This Lagrangian cycle $CC(F)\in H^0_\Lambda(T^*X;\pi_X^{-1}\omega_X)$ is called the characteristic cycle of F.

Although the definition of $CC(F)$ in [19] involves a very special bifunctor $\mu hom(\cdot,\cdot)$: $\mathbf{D}^b(X)^{\mathrm{op}}\times\mathbf{D}^b(X)\to\mathbf{D}^b(T^*X)$, here we follow the original approach by Kashiwara [17].

First, we define the sheaf $\mathscr{L}_X$ of Lagrangian cycles on T^*X by

$$\mathscr{L}_X=\varinjlim_{\Lambda} H^0R\Gamma_\Lambda\big(\pi_X^{-1}\omega_X\big),$$

where Λ ranges through the family L_X of closed conic subanalytic Lagrangian subsets of T^*X. Then for $\Lambda\in L_X$ we have an isomorphism

$$H^0_\Lambda\big(T^*X;\pi_X^{-1}\omega_X\big)\simeq\Gamma_\Lambda(T^*X;\mathscr{L}_X).$$

We are going to define $CC(F)$ as a section of $\mathscr{L}_X$ whose support is contained in the micro-support $\Lambda=SS(F)\subset T^*X$ of F. We take an open subanalytic subset $\Omega\subset T^*X$ such that $\Lambda\cap\Omega$ is an open dense manifold in Λ and for each connected component Λ_i of $\Lambda\cap\Omega=\bigsqcup_{i\in I}\Lambda_i$, there exists a submanifold $X_{\alpha_i}\subset X$ satisfying the condition $\Lambda_i\subset T^*_{X_{\alpha_i}}X$. Since for such $\Omega\subset T^*X$ we have $\dim(\Lambda\setminus\Omega)<\dim\Lambda$ and the restriction map

$$\Gamma_\Lambda(T^*X;\mathscr{L}_X)\to\Gamma_{\Lambda\cap\Omega}(\Omega;\mathscr{L}_X)$$

is injective, it suffices to specify the image of $CC(F)$ in $\Gamma_{\Lambda\cap\Omega}(\Omega;\mathscr{L}_X)$. Finally for a submanifold $Y\subset X$, we denote by $[T^*_YX]$ the fundamental cycle supported by the conormal bundle T^*_YX (see Definition 2.3 below).

Definition 2.1. [17] We define $CC(F)\in\Gamma_\Lambda(T^*X;\mathscr{L}_X)$ to be the unique section of $\mathscr{L}_X$ whose restriction to Ω satisfies the property

$$CC(F)=m_i[T^*_{X_{\alpha_i}}X]$$

in an open neighborhood of Λ_i in T^*X for any $i\in I$. Here m_i is an integer defined by

$$m_i=\sum_{j\in\mathbb{Z}}(-1)^j\dim H^j\big(R\Gamma_{\{\psi\geqslant\psi(x_{\alpha_i})\}}(F)_{x_{\alpha_i}}\big)$$

for a real analytic function $\psi : X \to \mathbb{R}$ on a neighborhood of a point $x_{\alpha_i} \in X_{\alpha_i}$ in X satisfying the conditions:

(a) $(x_{\alpha_i}; \operatorname{grad}\psi(x_{\alpha_i})) \in \Lambda_i \subset T^*_{X_{\alpha_i}} X$.
(b) The Hessian of $\psi|_{X_{\alpha_i}}$ at the point $x_{\alpha_i} \in X_{\alpha_i}$ is positive definite.

Next, we recall the functorial properties of characteristic cycles of constructible sheaves. Let $f : X \to Y$ be a morphism of real analytic manifolds and consider the natural maps

$$T^*X \underset{{}^t f'}{\longleftarrow} X \times_Y T^*Y \underset{f_\pi}{\longrightarrow} T^*Y$$

associated with it. Then for a closed conic subanalytic isotropic subset Λ_X of T^*X (respectively Λ_Y of T^*Y) such that f_π is proper on $({}^t f')^{-1}(\Lambda_X)$ (respectively ${}^t f'$ is proper on $f_\pi^{-1}(\Lambda_Y)$), there exists a natural morphism

$$f_* : H^0_{\Lambda_X}\big(T^*X; \pi_X^{-1}\omega_X\big) \to H^0_{f_\pi({}^t f')^{-1}(\Lambda_X)}\big(T^*Y; \pi_Y^{-1}\omega_Y\big)$$
$$\big(\text{respectively } f^* : H^0_{\Lambda_Y}\big(T^*Y; \pi_Y^{-1}\omega_Y\big) \to H^0_{{}^t f' f_\pi^{-1}(\Lambda_Y)}\big(T^*X; \pi_X^{-1}\omega_X\big)\big).$$

Proposition 2.2. [19, Propositions 9.4.2 and 9.4.3] *Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ (respectively $G \in \mathbf{D}^b_{\mathbb{R}-c}(Y)$) and assume that f is proper on* supp F *(respectively ${}^t f'$ is proper on $f_\pi^{-1}(SS(G))$). Then we have $CC(Rf_*F) = f_*CC(F)$ (respectively $CC(f^{-1}G) = f^*CC(G)$).*

Having these operations of Lagrangian cycles at hands, we are ready to define the fundamental cycle $[T^*_Y X] \in \Gamma_{T^*_Y X}(T^*X; \mathscr{L}_X)$ of the conormal bundle to a closed submanifold $Y \subset X$.

Definition 2.3. [19, Example 9.3.4]

(i) For the 0-dimensional manifold $\{\mathrm{pt}\}$, since we have $\mathscr{L}_{\{\mathrm{pt}\}} = \mathbb{C}$, we denote by $[\mathrm{pt}]$ the Lagrangian cycle in $T^*\{\mathrm{pt}\} \simeq \{\mathrm{pt}\}$ which corresponds to $1 \in \mathbb{C}$.
(ii) For a real analytic manifold X ($a_X : X \to \{\mathrm{pt}\}$), we set

$$[T^*_X X] = a_X^*[\mathrm{pt}] \in \Gamma_{T^*_X X}(T^*X; \mathscr{L}_X).$$

(iii) For a closed embedding of a real analytic manifold $f : Y \hookrightarrow X$, we set

$$[T^*_Y X] = f_*[T^*_Y Y] \in \Gamma_{T^*_Y X}(T^*X; \mathscr{L}_X).$$

In subsequent sections, the fundamental cycles $[T^*_Y X]$ of conormal bundles $T^*_Y X$ will play a crucial role in the study of real projective duality. For a closed submanifold $Y \subset X$, we consider the sheaf $\mathbb{C}_Y$ as an object in $\mathbf{D}^b_{\mathbb{R}-c}(X)$. Then by definition we have

$$CC(\mathbb{C}_Y) = [T^*_Y X].$$

Moreover the map

$$CC(\cdot) : \mathbf{D}^b_{\mathbb{R}-c}(X) \to \Gamma(T^*X; \mathscr{L}_X)$$

evidently satisfies the following properties.

(a) Let $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$. Then for any $k \in \mathbb{Z}$ we have

$$CC\bigl(F[k]\bigr) = (-1)^k CC(F).$$

(b) Let $F' \to F \to F'' \xrightarrow{+1}$ be a distinguished triangle in $\mathbf{D}^b_{\mathbb{R}-c}(X)$. Then

$$CC(F) = CC(F') + CC(F'').$$

Let us define the Grothendieck group $\mathbf{K}_{\mathbb{R}-c}(X)$ of $\mathbf{D}^b_{\mathbb{R}-c}(X)$ to be the quotient of the free abelian group generated by the objects in $\mathbf{D}^b_{\mathbb{R}-c}(X)$ by the relations

$$F = F' + F'' \quad \bigl(F' \to F \to F'' \xrightarrow{+1} \text{ is a distinguished triangle}\bigr).$$

Then by the property (b) above, we obtain a group homomorphism

$$CC(\cdot) : \mathbf{K}_{\mathbb{R}-c}(X) \to \Gamma(T^*X; \mathscr{L}_X).$$

This morphism is in fact an isomorphism. See [19, Chapter IX] for the proof.

2.2. Constructible functions

We shall introduce the calculus of constructible functions developed by Kashiwara–Schapira [19], Schapira [26] and Viro [33].

Let X be a real analytic manifold.

Definition 2.4. We say that a $\mathbb{Z}$-valued function $\varphi : X \to \mathbb{Z}$ is constructible if it satisfies the following equivalent conditions:

(i) There exists a stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X by subanalytic submanifolds X_α's such that $\varphi|_{X_\alpha}$ is constant for each $\alpha \in A$.
(ii) There exists a locally finite family of compact subanalytic subsets $\{K_\beta\}_{\beta \in B}$ such that

$$\varphi = \sum_{\beta \in B} m_\beta \mathbf{1}_{K_\beta}.$$

Here $m_\beta \in \mathbb{Z}$ and $\mathbf{1}_Z$ is the characteristic function of the subset Z.

We set $CF(X) = \{\varphi : X \to \mathbb{Z} \mid \varphi \text{ is constructible}\}$.

For $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$, the function $\chi(F) : X \to \mathbb{Z}$ defined by taking the local Euler–Poincaré index

$$\chi(F)(x) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(F_x)$$

at each $x \in X$ is constructible. This induces a group homomorphism

$$\chi : \mathbf{K}_{\mathbb{R}-c}(X) \to CF(X). \tag{2.1}$$

Theorem 2.5. [19, Theorem 9.7.1] χ *is an isomorphism.*

We recall basic operations of constructible functions. These operations are induced by those of constructible sheaves through this Euler–Poincaré index χ.

Definition 2.6. Let X and Y be real analytic manifolds and $f : X \to Y$ a real analytic map.

(i) For a constructible function $\varphi \in CF(X)$ with compact support, we define the Euler (topological) integral of φ by

$$\int_X \varphi = \chi(X; F) \in \mathbb{Z},$$

where $F \in \mathbf{K}_{\mathbb{R}-c}(X)$ such that $\chi(F) = \varphi$. This Euler integral $\int_X \varphi$ can be calculated more easily as follows. If $\varphi = \sum_{\alpha \in A} m_\alpha \mathbf{1}_{K_\alpha}$ for a family of locally finite compact subanalytic subsets $\{K_\alpha\}_{\alpha \in A}$ as in Definition 2.4(ii), then we have

$$\int_X \varphi = \sum_{\alpha \in A} m_\alpha \chi(K_\alpha).$$

(ii) Let $\varphi \in CF(X)$. Assume that $f : \mathrm{supp}(\varphi) \to Y$ is proper. We define the direct image $\int_f \varphi \in CF(Y)$ of φ by

$$\left(\int_f \varphi\right)(y) = \int_X \varphi \cdot \mathbf{1}_{f^{-1}(y)}.$$

Note that if moreover X is compact, the following diagram is commutative.

$$\begin{tikzcd} \mathbf{K}_{\mathbb{R}-c}(X) \arrow[r, "Rf_*"] \arrow[d, "\chi"', "\wr"] & \mathbf{K}_{\mathbb{R}-c}(Y) \arrow[d, "\wr"', "\chi"] \\ CF(X) \arrow[r, "\int_f"'] & CF(Y). \end{tikzcd}$$

(iii) Let $\psi \in CF(Y)$. We define the inverse image $f^*\psi \in CF(X)$ of ψ by

$$(f^*\psi)(x) = \psi\bigl(f(x)\bigr).$$

Clearly this operation is also compatible with the one $f^{-1} : \mathbf{K}_{\mathbb{R}-c}(Y) \to \mathbf{K}_{\mathbb{R}-c}(X)$ for constructible sheaves.

Definition 2.7. Let $\varphi \in CF(X)$. We define the characteristic cycle of φ by

$$CC(\varphi) = CC(F),$$

where $F \in \mathbf{K}_{\mathbb{R}-c}(X)$ such that $\chi(F) = \varphi$.

3. Microlocal topological Radon transforms

In this section, we present some basic results on the topological Radon transforms of Lagrangian cycles. It seems that the corresponding results for constructible sheaves are implicitly known to specialists (in the real analytic setting especially after Kashiwara–Schapira [19]). However, it is hard to find them explicitly in the literature and moreover they were not yet proved directly in the framework of Lagrangian cycles as in Proposition 3.4.

Let X and Y be two real analytic manifolds and S a closed submanifold of $X \times Y$. We consider the following commutative diagram:

$$\begin{tikzcd} & X \times Y \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\ & S \arrow[u, hook] \arrow[dl, "f"] \arrow[dr, "g"'] & \\ X & & Y, \end{tikzcd} \tag{3.1}$$

where p_1 and p_2 are the first and second projections respectively. In this setting, the topological Radon transform $\mathcal{R}_S(\varphi)$ of a constructible function $\varphi \in CF(X)$ on X is defined by

$$\mathcal{R}_S(\varphi) = \int_g f^*\varphi = \int_{p_2} \mathbf{1}_S \cdot p_1^*\varphi.$$

Then we obtain a group homomorphism $\mathcal{R}_S : CF(X) \to CF(Y)$.

In the special case where

$$X = \mathbb{P}_n = \left\{ l \subset \mathbb{R}^{n+1} \mid l \text{ is a real line s.t. } 0 \in l \right\}$$

(the n-dimensional real projective space),

$$Y = \mathbb{G}_{n,k} = \left\{ L \subset \mathbb{R}^{n+1} \mid L \text{ is a } (k+1)\text{-dimensional linear subspace in } \mathbb{R}^{n+1} \right\}$$

(the real Grassmann manifold $(1 \leqslant k \leqslant n-1)$) and

$$S = \left\{ (l, L) \in X \times Y \mid 0 \in l \subset L \subset \mathbb{R}^{n+1} \right\}$$

(the incidence submanifold), topological Radon transforms were studied by many mathematicians (see, for example, [4,5,8,21,27,33], etc.). In order to study them microlocally, we follow the approach in [6].

We come back to the general setting. Assume that f (respectively g) is a smooth (respectively smooth proper) morphism. Then we obtain natural morphisms

$$T^*X \underset{f_\pi}{\twoheadleftarrow} S \times_X T^*X \underset{{}^tf'}{\hookrightarrow} T^*S,$$

$$T^*Y \underset{g_\pi}{\twoheadleftarrow} S \times_Y T^*Y \underset{{}^tg'}{\hookrightarrow} T^*S.$$

Therefore we may consider $S \times_X T^*X$ and $S \times_Y T^*Y$ as closed submanifolds of T^*S. We shall describe the intersection $(S \times_X T^*X) \cap (S \times_Y T^*Y)$ in T^*S using the conormal bundle $T^*_S(X \times Y) \subset T^*(X \times Y)$. First, let us set

$$\Delta_g = \{(s, y) \in S \times Y \mid y = g(s)\} \subset S \times Y.$$

Note that Δ_g is the image of the graph embedding $\mathrm{id}_S \times g : S \hookrightarrow S \times Y$ of S. Similarly we set

$$\widetilde{\Delta_f} = \{(x, s) \in X \times S \mid x = f(s)\} \subset X \times S.$$

Then we get natural isomorphisms

$$S \times_X T^*X \simeq T^*_{\widetilde{\Delta_f}}(X \times S),$$

$$S \times_Y T^*Y \simeq T^*_{\Delta_g}(S \times Y).$$

If we consider the fiber product $T^*_{\widetilde{\Delta_f}}(X \times S) \times_{T^*S} T^*_{\Delta_g}(S \times Y)$ defined by the maps

$$T^*_{\widetilde{\Delta_f}}(X \times S) \hookrightarrow T^*X \times T^*S \underset{p_2}{\longrightarrow} T^*S,$$

$$T^*_{\Delta_g}(S \times Y) \hookrightarrow T^*S \times T^*Y \underset{p_1}{\longrightarrow} T^*S,$$

then we have $T^*_{\widetilde{\Delta_f}}(X \times S) \times_{T^*S} T^*_{\Delta_g}(S \times Y) \subset T^*X \times T^*S \times T^*Y$. Denote by α_{T^*Y} the antipodal map of the cotangent bundle T^*Y. The following result was found by D'Agnolo–Schapira [6, p. 363].

Lemma 3.1. [6] *Let p_{13} be the projection $T^*X \times T^*S \times T^*Y \to T^*X \times T^*Y$. Then p_{13} induces a closed embedding $T^*_{\widetilde{\Delta_f}}(X \times S) \times_{T^*S} T^*_{\Delta_g}(S \times Y) \hookrightarrow T^*X \times T^*Y$. Moreover the image of this map is* $(\mathrm{id}_{T^*X} \times \alpha_{T^*Y})(T^*_S(X \times Y)) \subset T^*(X \times Y)$.

Hence we finally obtain the identification

$$(S \times_X T^*X) \cap (S \times_Y T^*Y) \simeq (\mathrm{id}_{T^*X} \times \alpha_{T^*Y})\big(T^*_S(X \times Y)\big) =: T^*_S(X \times Y)^a.$$

In what follows, we naturally identify this $T_S^*(X\times Y)^a$ with $T_S^*(X\times Y)$. Also, we shall frequently use the diagram

$$\begin{tikzcd}
T^*X & S\times_X T^*X \arrow[l, two heads, "f_\pi"'] \arrow[d, hook, "{}^t f'"'] \arrow[dr, phantom, "\square"] & T_S^*(X\times Y) \arrow[l, hook', "h"'] \arrow[d, hook, "k"] \\
 & T^*S & S\times_Y T^*Y \arrow[l, hook', "{}^t g'"] \arrow[d, "g_\pi"] \\
 & & T^*Y,
\end{tikzcd}$$

whose upper-right square is of Cartesian.

Proposition 3.2. *Assume that the intersection of $S\times_X T^*X$ and $S\times_Y T^*Y$ in T^*S is transversal. Then there exists an isomorphism*

$$h^{-1}f_\pi^{-1}\left(\pi_X^{-1}\omega_X\right)\simeq k^! g_\pi^!\left(\pi_Y^{-1}\omega_Y\right)$$

in the derived category $\mathbf{D}^b(T_S^*(X\times Y))$.

Proof. First recall that the cotangent bundles T^*X, T^*S and T^*Y are orientable. We give them the standard orientations ε_{T^*X}, ε_{T^*S} and ε_{T^*Y} respectively. If $(x_1,\dots,x_n;\xi_1,\dots,\xi_n)$ is a local coordinate system of T^*X, then this orientation ε_{T^*X} is locally defined by the form $dx_1\wedge\cdots\wedge dx_n\wedge d\xi_1\wedge\cdots\wedge d\xi_n$. Consequently there exist natural isomorphisms $\pi_X^{-1}\omega_X\simeq \pi_X^!\mathbb{C}_X=\omega_{T^*X/X}$, $\pi_S^{-1}\omega_S\simeq\omega_{T^*S/S}$, etc. Therefore we get a chain of isomorphisms

$$\begin{aligned}
h^{-1}f_\pi^{-1}\left(\pi_X^{-1}\omega_X\right) &\simeq h^{-1}f_\pi^{-1}\omega_{T^*X/X}\simeq h^{-1}\omega_{S\times_X T^*X/S}\simeq h^{-1}\left({}^t f'\right)^!\omega_{T^*S/S} \\
&\xrightarrow{\sim} k^!\left({}^t g'\right)^{-1}\omega_{T^*S/S}.
\end{aligned}$$

To show the last isomorphism, we used the transversality of the intersection $(S\times_X T^*X)\cap(S\times_Y T^*Y)$ and the fact that the cohomology sheaves of $\omega_{T^*S/S}$ are locally constant.

Let us consider the commutative diagram

$$\begin{tikzcd}
T^*S \arrow[dr, "\pi_S"'] & S\times_Y T^*Y \arrow[l, hook', "{}^t g'"'] \arrow[r, two heads, "g_\pi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & T^*Y \arrow[d, "\pi_Y"] \\
 & S \arrow[r, two heads, "g"'] & Y.
\end{tikzcd}$$

Then we finally get

$$k^!\left({}^t g'\right)^{-1}\omega_{T^*S/S}\simeq k^!\left({}^t g'\right)^{-1}\pi_S^{-1}\omega_S\simeq k^!\pi^{-1}\omega_S=k^!\pi^{-1}g^!\omega_Y\simeq k^! g_\pi^!\left(\pi_Y^{-1}\omega_Y\right).$$

This completes the proof. $\square$

Consider the diagram

$$\begin{tikzcd} & T_S^*(X \times Y) \arrow[dl, "\overline{p_1}"'] \arrow[dr, "\overline{p_2}^a"] & \\ T^*X & & T^*Y, \end{tikzcd}$$

where $\overline{p_1}$ and $\overline{p_2}^a$ are induced by the projections

$$p_1 : T^*X \times T^*Y \to T^*X,$$
$$p_2^a : T^*X \times T^*Y \xrightarrow[p_2]{} T^*Y \xrightarrow[\alpha_{T^*Y}]{\sim} T^*Y$$

(α_{T^*Y} is the antipodal map of T^*Y). Note that p_2^a is a natural projection from $T_S^*(X \times Y)^a$ to T^*Y. Therefore we have $\overline{p_1} = f_\pi \circ h$, $\overline{p_2}^a = g_\pi \circ k$ and Proposition 3.2 implies the isomorphism $\overline{p_1}^{-1}(\pi_X^{-1}\omega_X) \simeq \overline{p_2}^{a\,!}(\pi_Y^{-1}\omega_Y)$.

Now, let us consider the case where $X = \mathbb{P}_n$, $Y = \mathbb{G}_{n,k}$ for $1 \leqslant k \leqslant n-1$ and S is the incidence manifold. In this situation, the following result is well known:

Proposition 3.3. ([4, Lemma 5.6] and [12]) *Let $X = \mathbb{P}_n$ and $Y = \mathbb{G}_{n,k}$ for $1 \leqslant k \leqslant n-1$. Then*

(i) *Outside the zero-sections, $\overline{p_1}$ (respectively $\overline{p_2}^a$) induces a smooth map*

$$\widetilde{p_1} : \dot{T}_S^*(X \times Y) \twoheadrightarrow \dot{T}^*X$$

(respectively a closed embedding $\widetilde{p_2}^a : \dot{T}_S^*(X \times Y) \hookrightarrow \dot{T}^*Y$).

(ii) *If moreover $k = n-1$ (i.e. if Y is the dual projective space $\mathbb{P}_n^*$ of $X = \mathbb{P}_n$), then these maps are isomorphisms*

$$\begin{tikzcd} & \dot{T}_S^*(X \times Y) \arrow[dl, "\widetilde{p_1}"', "\sim"] \arrow[dr, "\widetilde{p_2}^a", "\sim"'] & \\ \dot{T}^*X & & \dot{T}^*Y. \end{tikzcd}$$

For the reader's convenience, we shall briefly explain the outline of the proof of this proposition. First, recall that $X = \mathbb{P}_n$ is a homogeneous space of the real Lie group $G = SL_{n+1}(\mathbb{R})$ by the natural action $G \curvearrowright X$. From this, we obtain a moment map

$$T^*X \to \mathfrak{g}^* = (\operatorname{Lie} G)^*.$$

Using the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ defined by the Killing form of $\mathfrak{g}$, this moment map yields a commutative diagram

$$\begin{tikzcd} T^*X \arrow[rr, hook] \arrow[dr, "\pi_X"'] & & X \times \mathfrak{g} \arrow[dl, "p_1"] \\ & X, & \end{tikzcd}$$

in which the horizontal arrow $T^*X \hookrightarrow X \times \mathfrak{g}$ is injective. If we identify the cotangent bundle T^*X with its image in $X \times \mathfrak{g}$ ($\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{R}) \subset \mathfrak{gl}_{n+1}(\mathbb{R})$), then we obtain the following handy description of T^*X.

$$T^*X = T^*\mathbb{P}_n \simeq \{(l, A) \in \mathbb{P}_n \times \mathfrak{gl}_{n+1}(\mathbb{R}) \mid \operatorname{Im} A \subset l \subset \operatorname{Ker} A\}.$$

Similarly we have

$$T^*Y \simeq \{(L, A) \in \mathbb{G}_{n,k} \times \mathfrak{gl}_{n+1}(\mathbb{R}) \mid \operatorname{Im} A \subset L \subset \operatorname{Ker} A\},$$
$$T^*_S(X \times Y) \simeq \{(l, L, A) \in \mathbb{P}_n \times \mathbb{G}_{n,k} \times \mathfrak{gl}_{n+1}(\mathbb{R}) \mid \operatorname{Im} A \subset l \subset L \subset \operatorname{Ker} A\}.$$

Moreover $\widetilde{p_1}$ (respectively $\widetilde{p_2}^a$) is given simply by $(l, L, A) \mapsto (l, A)$ (respectively $(l, L, A) \mapsto (L, -A)$), from which Proposition 3.3 immediately follows.

Using Proposition 3.2, we can easily prove the following result on the Radon transforms of characteristic cycles.

Proposition 3.4. *Let $X = \mathbb{P}_n$, $Y = \mathbb{G}_{n,k}$ for $1 \leqslant k \leqslant n-1$ and consider the diagram* (3.1). *Then the composite of the natural maps*

$$\Gamma(T^*X; \mathscr{L}_X) \underset{f^*}{\longrightarrow} \Gamma(T^*S; \mathscr{L}_S) \underset{g_*}{\longrightarrow} \Gamma(T^*Y; \mathscr{L}_Y)$$

of Lagrangian cycles induces an isomorphism

$$\Gamma_{\Lambda_X}(\dot{T}^*X; \mathscr{L}_X) \xrightarrow{\sim} \Gamma_{\Lambda_Y}(\dot{T}^*Y; \mathscr{L}_Y)$$

for any closed conic subanalytic Lagrangian subset $\Lambda_X \subset \dot{T}^*X$ and $\Lambda_Y = \widetilde{p_2}^a \widetilde{p_1}^{-1} \Lambda_X$. In particular, if $k = n-1$ (i.e. if $Y = \mathbb{P}_n^*$), we obtain an isomorphism

$$\Gamma(\dot{T}^*X; \mathscr{L}_X) \xrightarrow{\sim} \Gamma(\dot{T}^*Y; \mathscr{L}_Y).$$

Proof. By the above handy descriptions of $T^*_S(X \times Y)$, T^*X and T^*Y, etc., we see that for any $s \in S$ the fibers of the vector bundles $S \times_X T^*X$ and $S \times_Y T^*Y$ at s intersects transversally in $T^*_s S$. This implies that the hypothesis of Proposition 3.2 is satisfied. Hence we get

$$\widetilde{p_1}^{-1}(\pi_X^{-1}\omega_X) \simeq \widetilde{p_2}^{a!}(\pi_Y^{-1}\omega_Y).$$

We need the following elementary lemma.

Lemma 3.5. *Let $p : M \twoheadrightarrow N$ be a smooth surjective morphism of real analytic manifolds and $Z \subset N$ a closed subanalytic subset of codimension d. Assume that all the fibers of p are connected. Then the natural morphism*

$$H_Z^d(N; \mathscr{L}) \to H_{p^{-1}(Z)}^d\big(M; p^{-1}\mathscr{L}\big)$$

is an isomorphism for any local system $\mathscr{L}$ on N.

By this lemma, we have formally the chain of isomorphisms

$$\begin{aligned}
\Gamma_{\Lambda_X}(\dot{T}^*X; \mathscr{L}_X) &\simeq H_{\Lambda_X}^0\big(\dot{T}^*X; \pi_X^{-1}\omega_X\big) \\
&\xrightarrow{\sim} H^0_{\widetilde{p_1}^{-1}(\Lambda_X)}\big(\dot{T}_S^*(X \times Y); \widetilde{p_1}^{-1}\pi_X^{-1}\omega_X\big) \\
&\xrightarrow{\sim} H^0_{\widetilde{p_1}^{-1}(\Lambda_X)}\big(\dot{T}_S^*(X \times Y); \widetilde{p_2}^{a\,!}\pi_Y^{-1}\omega_Y\big) \\
&\simeq H^0_{\widetilde{p_2}^a\widetilde{p_1}^{-1}(\Lambda_X)}\big(\dot{T}^*Y; \pi_Y^{-1}\omega_Y\big) \\
&\simeq \Gamma_{\Lambda_Y}(\dot{T}^*Y; \mathscr{L}_Y).
\end{aligned}$$

To finish the proof, we have to show that this isomorphism is induced by the morphisms

$$\Gamma(T^*X; \mathscr{L}_X) \underset{f^*}{\to} \Gamma(T^*S; \mathscr{L}_S) \underset{g_*}{\to} \Gamma(T^*Y; \mathscr{L}_Y)$$

of Lagrangian cycles (see [19, Proposition 9.3.2]). It is straightforward to show this. $\square$

Now denote by Σ the image of $\dot{T}_S^*(X \times Y)$ by $\widetilde{p_2}^a$. We know that Σ is a regular involutive submanifold of $\dot{T}^*Y$. Along the same line as Proposition 3.4, we can prove the following stronger result.

Corollary 3.6. *Let $\Omega_X \subset \dot{T}^*X$ be an open subset and $\Lambda_X \subset \Omega_X$ a closed conic subanalytic Lagrangian subset. We also take an open subset $\Omega_Y \subset \dot{T}^*Y$ such that $\widetilde{p_1}^{-1}(\Omega_X) = \widetilde{p_2}^{a\;-1}(\Omega_Y)$. Then for $\Lambda_Y = \widetilde{p_2}^a\widetilde{p_1}^{-1}(\Lambda_X) \subset \Sigma \cap \Omega_Y$, we have an isomorphism*

$$\Gamma_{\Lambda_X}(\Omega_X; \mathscr{L}_X) \simeq \Gamma_{\Lambda_Y}(\Omega_Y; \mathscr{L}_Y).$$

In particular, $\Gamma(\Omega_X; \mathscr{L}_X) \xrightarrow{\sim} \Gamma_{\Sigma\cap\Omega_Y}(\Omega_Y; \mathscr{L}_Y)$.

As an immediate consequence of Proposition 3.4, we also have the following result. For a Lagrangian cycle λ, we denote its support by $|\lambda|$.

Corollary 3.7. *Let $\varphi \in CF(X)$. Then the Radon transform $\mathcal{R}_S(\varphi)$ of φ satisfies*

$$\big|CC\big(\mathcal{R}_S(\varphi)\big)\big| = \widetilde{p_2}^a\widetilde{p_1}^{-1}\big|CC(\varphi)\big|$$

in $\dot{T}^*Y$.

4. Curvatures and sign changes of characteristic cycles

In this section, we focus our attention on topological Radon transforms between real projective spaces $X=\mathbb{P}_n$ and their duals $Y=\mathbb{P}_n^*$. We describe the sign changes of characteristic cycles of constructible functions by the topological Radon transform $\mathcal{R}_S : CF(X) \to CF(Y)$ in terms of curvatures of strata in X. Namely, we show in Theorem 4.3 that, in order to write down the characteristic cycle of the Radon transform of $\mathcal{R}_S(\varphi)$ of a constructible function $\varphi \in CF(X)$, it suffices to know the signatures of curvatures of strata in the stratification $X=\bigsqcup_{\alpha\in A} X_\alpha$ adapted to φ. This relevance of signatures of curvatures is peculiar to the real setting. Compare this with the results of Ernström [8] in the complex case.

Using the notations in previous sections, first recall the isomorphism

$$\mathcal{R}_S^\mu = g_* \circ f^* : \Gamma(\dot{T}^*X; \mathscr{L}_X) \xrightarrow{\sim} \Gamma(\dot{T}^*Y; \mathscr{L}_Y)$$

obtained in Proposition 3.4. Let us call this isomorphism the microlocal Radon transform. If we set

$$\Phi := \left(\widetilde{p_2}^a \circ \widetilde{p_1}^{-1}\right) : \dot{T}^*X \xrightarrow{\sim} \dot{T}^*Y,$$

then it follows also from Proposition 3.4 that

$$\left|CC\left(\mathcal{R}_S(\varphi)\right)\right| = \Phi\left(\left|CC(\varphi)\right|\right)$$

in $\dot{T}^*Y$ for any $\varphi \in CF(X)$. This result implies that the characteristic cycle $CC(\mathcal{R}_S(\varphi))$ of the topological Radon transform $\mathcal{R}_S(\varphi)$ of $\varphi \in CF(X)$ can be described by that of φ outside the zero sections. Our aim here is to give such a description. In Section 5, we apply this result to the study of real projective duality.

4.1. Reduction of the problem

To begin with, let us fix a μ-stratification (for the definition see [19, Chapter VIII]) $X=\bigsqcup_{\alpha\in A} X_\alpha$ of X adapted to the given constructible function $\varphi \in CF(X)$. Then $\bigsqcup_{\alpha\in A} T^*_{X_\alpha}X \subset T^*X$ is a closed conic subanalytic Lagrangian subset and for $\Lambda := |CC(\varphi)|$ we have $\Lambda \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X$. In this situation, we can take an open dense subanalytic subset $\Omega \subset \dot{T}^*X = T^*X \setminus T^*_X X$ such that

(i) $\Lambda \cap \Omega$ is a smooth manifold and dense in $\Lambda \cap \dot{T}^*X$.
(ii) Let $\Lambda \cap \Omega = \bigsqcup_{i\in I} \Lambda_i$ be the decomposition of $\Lambda \cap \Omega$ into connected components. Then for any $i \in I$, there exists $\alpha_i \in A$ such that $\Lambda_i \subset \dot{T}^*_{X_{\alpha_i}}X$.

Our task is to describe explicitly the microlocal Radon transform

$$\mathcal{R}_S^\mu\left(CC(\varphi)|_{\Lambda_i}\right) = CC\left(\mathcal{R}_S(\varphi)\right)\big|_{\Phi(\Lambda_i)}$$

of $CC(\varphi)|_{\Lambda_i}$ for each $i \in I$.

Fix $i \in I$ and take $\alpha_i \in A$ such that $\Lambda_i \subset \dot{T}^*_{X_{\alpha_i}} X$. Since

$$CC(\varphi)|_{\Lambda_i} = m_i [T^*_{X_{\alpha_i}} X]|_{\Lambda_i} \quad \text{for some integer } m_i \in \mathbb{Z},$$

it suffices to study the microlocal Radon transform of $CC(\mathbf{1}_{X_{\alpha_i}})|_{\Lambda_i}$. Therefore, assuming that we are given a locally closed connected subanalytic submanifold M of $X = \mathbb{P}_n$ such that $\dim M < n = \dim X$, we shall study the Radon transform of $\mathbf{1}_M \in CF(X)$.

Since we have $|CC(\mathcal{R}_S(\mathbf{1}_M))| = \Phi(\dot{T}^*_M X)$ in $\dot{T}^*Y$, we first study the case where $\Phi(\dot{T}^*_M X)$ is contained in the conormal bundle $\dot{T}^*_N Y$ of a locally closed submanifold $N \subset Y$.

4.2. *Curvatures of submanifolds in real projective spaces*

For our purpose, we here introduce a Riemannian metric of $X = \mathbb{P}_n$ as follows. We consider the natural surjective map

$$\kappa : S^n \twoheadrightarrow X = \mathbb{P}_n \simeq S^n/\{\pm 1\}$$

from the unit n-sphere S^n in $\mathbb{R}^{n+1}$. Since the metric of S^n induced from the Euclidean one of $\mathbb{R}^{n+1}$ is invariant by the antipodal map of S^n, there exists a unique metric of $\mathbb{P}_n$ which comes down from that of S^n. This is the canonical metric of $X = \mathbb{P}_n$ and for $M \subset X$ we consider the metric induced from X.

Now for two C^∞-vector fields $\vec{u}, \vec{v}$ on M, we denote by $\nabla^X_{\vec{u}} \vec{v}$ (respectively $\nabla^M_{\vec{u}} \vec{v}$) the covariant derivative of $\vec{v}$ by $\vec{u}$ with respect to the canonical metric of X (respectively with respect to the induced metric of M). These two covariant derivatives are different in general. The difference will be expressed by the second fundamental form that we introduce below.

Let $(TM)^\perp$ be the subbundle of $M \times_M TX$ consisting of tangent vectors orthogonal to M. Then $(TM)^\perp$ is isomorphic to the normal bundle $T_M X$ of M in X and we have

$$M \times_M TX \simeq TM \oplus (TM)^\perp.$$

From this, we obtain a decomposition $\nabla^X_{\vec{u}} \vec{v} = \vec{w}_1 + \vec{w}_2$ of $\nabla^X_{\vec{u}} \vec{v} \in M \times_M TX$ such that $\vec{w}_1 \in TM$ and $\vec{w}_2 \in (TM)^\perp$. It is well known in differential geometry that $\vec{w}_1$ is equal to $\nabla^M_{\vec{u}} \vec{v}$. Moreover by the correspondence

$$(\vec{u}, \vec{v}) \mapsto \vec{w}_2,$$

we get a well-defined symmetric bilinear form

$$h_M(\cdot,\cdot)_x : T_x M \times T_x M \to (TM)^\perp_x \simeq (T_M X)_x$$

at each $x \in M$. Thus a map

$$h_M(\cdot,\cdot) : TM \times TM \to T_M X$$

is obtained, which is usually called the second fundamental form of M in X.

For a covector $\theta_x \in (T^*_M X)_x \subset T^*_M X$, let us define a symmetric bilinear form

$$h_{M,\theta_x}(\cdot,\cdot) : T_x M \times T_x M \to \mathbb{R}$$

by $h_{M,\theta_x}(\vec{u}_x, \vec{v}_x) = \langle \theta_x, h_M(\vec{u}_x, \vec{v}_x)\rangle$ for $\vec{u}_x, \vec{v}_x \in T_xM$ and set

$$\mathcal{Z}_{\theta_x} = \left\{ \vec{u}_x \in T_xM \mid h_{M,\theta_x}(\vec{u}_x, \vec{v}_x) = 0 \text{ for any } \vec{v}_x \in T_xM \right\} \subset T_xM.$$

We call $\mathcal{Z}_{\theta_x}$ the space of nullity of the second fundamental form h_{M,θ_x}. If we define a non-negative integer $r \geqslant 0$ by $r = \min\{\dim \mathcal{Z}_{\theta_x} \mid \theta_x \in \dot{T}^*_M X\}$, then there exists an open dense subset $\Omega_0 \subset \dot{T}^*_M X$ such that $\dim \mathcal{Z}_{\theta_x} = r$ for any $\theta_x \in \Omega_0$ (use the fact that M is connected). The following result plays an important role in the study of projective duality of algebraic varieties.

Proposition 4.1. (Fisher–Piontkowski [9, Proposition 2.5.3]) *Let $\theta_x \in \dot{T}^*_M X$. Then the rank of the map*

$$\dot{\pi}_Y \circ \Phi : \dot{T}^*_M X \to Y = \mathbb{P}^*_n$$

at θ_x is $(n-1) - \dim \mathcal{Z}_{\theta_x}$.

As a consequence, there exist an open dense subanalytic subset $\Omega_1 \subset \dot{T}^*_M X$ and a locally closed (but not necessarily connected) subanalytic submanifold N of $Y = \mathbb{P}^*_n$ of dimension $(n-1) - r$ such that $\Phi(\Omega_1) \subset \dot{T}^*_N Y$.

Although Fisher–Piontkowski proved this proposition in the complex algebraic setting, the proof for the real case proceeds completely in the same way as in Proposition 2.5.3 of [9].

Note that the definitions of the second fundamental forms, etc. in [9] are slightly different from ours. For the reader's convenience, we shall quickly review the formulations of [9] and give the relation with ours.

In [9], instead of introducing the canonical metric of $X = \mathbb{P}_n$ and considering the curvature of M itself, Fisher and Piontkowski used the cone $\hat{M} \subset \mathbb{R}^{n+1} \setminus \{0\}$ over $M \subset \mathbb{P}_n$. Namely consider the commutative diagram

$$\begin{tikzcd} \hat{M} \arrow[r, hook] \arrow[d, "\tau_M"'] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "\tau_X"] \\ M \arrow[r, hook] & \mathbb{P}_n = X \end{tikzcd}$$

with τ_M and τ_X being smooth. For $x \in M$, take a point $q \in \hat{M}$ such that $\tau_M(q) = x$. Then by differentiating the Gauss map

$$\gamma_{\hat{M}} : \hat{M} \to \mathbb{G}_{n,\dim M}$$

($\hat{M} \ni p \mapsto T_p\hat{M} \subset T_p\mathbb{R}^{n+1} \xrightarrow{\sim} \mathbb{R}^{n+1}$) at $q \in \hat{M}$, we obtain a linear map

$$\gamma'_{\hat{M},q} : T_q\hat{M} \to \mathrm{Hom}_{\mathbb{R}}(T_q\hat{M}, T_q\mathbb{R}^{n+1}/T_q\hat{M})$$

and hence a symmetric bilinear form

$$\mathbb{I}_q : T_q\hat{M} \times T_q\hat{M} \to T_q\mathbb{R}^{n+1}/T_q\hat{M}.$$

This is the second fundamental form used in [9]. Note that this bilinear form is zero on $\mathbb{R}q \times \mathbb{R}q \subset T_q\hat{M} \times T_q\hat{M}$, where we identify the line $\mathbb{R}q$ in $\mathbb{R}^{n+1}$ with a subspace of $T_q\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$. As a result, we get another symmetric bilinear form

$$\check{\mathbb{I}}_q : T_q\hat{M}/\mathbb{R}q \times T_q\hat{M}/\mathbb{R}q \to T_q\mathbb{R}^{n+1}/T_q\hat{M}. \tag{4.1}$$

On the other hand, there exist natural isomorphisms

$$T_q\hat{M}/\mathbb{R}q \xrightarrow{\sim} T_xM,$$
$$T_q\mathbb{R}^{n+1}/T_q\hat{M} \xrightarrow{\sim} T_xX/T_xM \simeq (T_MX)_x$$

induced by the linear maps $\tau'_{M,q} : T_q\hat{M} \twoheadrightarrow T_xM$ and $\tau'_{X,q} : T_q\mathbb{R}^{n+1} \twoheadrightarrow T_xX$. Therefore the bilinear form in (4.1) can be rewritten as

$$\Psi_q : T_xM \times T_xM \to (T_MX)_x.$$

By [9, Lemma 2.4.1], we see that this bilinear form Ψ_q does not depend on the choice of the point $q \in \hat{M}$ over $x \in M$. Moreover, by an elementary differential calculus, we can easily prove that Ψ_q coincides with our second fundamental form

$$h_M(\cdot,\cdot)_x : T_xM \times T_xM \to (T_MX)_x$$

at $x \in M$.

4.3. Sign changes of characteristic cycles

Using the second fundamental form above, we describe the sign changes of the characteristic cycles of constructible functions by the topological Radon transform $\mathcal{R}_S$.

We recall the setting. Let M be a closed connected subanalytic submanifold of $X = \mathbb{P}_n$ such that $\dim M < n = \dim X$, $r = \min\{\dim \mathcal{Z}_{\theta_x} \mid \theta_x \in \dot{T}^*_MX\}$. Then there exist an open dense subanalytic subset $\Omega_1 \subset \dot{T}^*_MX$ and a locally closed subanalytic submanifold $N \subset Y = \mathbb{P}_n$ satisfying $\dim N = (n-1) - r$ and $\Phi(\Omega_1) \subset \dot{T}^*_NY$.

Definition 4.2. For $\theta_x \in \dot{T}^*_MX$, we denote by $I_M(\theta_x)$ the number (counted with multiplicities) of non-positive eigenvalues of the second fundamental form

$$h_{M,\theta_x}(\cdot,\cdot) : T_xM \times T_xM \to \mathbb{R}.$$

Now, let us state the main result of this section by using this $I_M(\theta_x)$.

Theorem 4.3. *Let $M \subset X$, $\Omega_1 \subset \dot{T}^*_MX$, $N \subset Y$ be as above. Then for any $\theta_x \in \Omega_1 \subset \dot{T}^*_MX$, we have the equality*

$$\mathcal{R}^{\mu}_S\big([T^*_MX]\big) = (-1)^{I_M(\theta_x)}[T^*_NY]$$

in an open neighborhood of $\Phi(\theta_x) \in \dot{T}^*_NY$.

Proof. First, we define affine open subsets of X and Y by

$$U_X = \{[a_0 : \cdots : a_n] \in X \mid a_0 \neq 0\} \subset X,$$
$$U_Y = \{[b_0 : \cdots : b_n] \in Y \mid b_n \neq 0\} \subset Y.$$

As usual, we take a coordinate system $(x_1, \dots, x_n)$ of U_X (respectively $(y_0, \dots, y_{n-1})$ of U_Y) given by $x_1 = a_1/a_0, x_2 = a_2/a_0, \dots, x_n = a_n/a_0$ (respectively $y_0 = b_0/b_n, y_1 = b_1/b_n, \dots, y_{n-1} = b_{n-1}/b_0$) and use the identification $U_X \simeq \mathbb{R}^n_{x_1,\dots,x_n}$ (respectively $U_Y \simeq \mathbb{R}^n_{y_0,\dots,y_{n-1}}$). Then the incidence submanifold $S \subset X \times Y$ can be written in the affine open subset $U_X \times U_Y \subset X \times Y$ in the following way.

$$S \cap (U_X \times U_Y) = \left\{ (x_1, \dots, x_n, y_0, \dots, y_{n-1}) \;\middle|\; y_0 + \sum_{i=1}^{n-1} x_i y_i + x_n = 0 \right\}.$$

Using the coordinate systems $(x; \xi) = (x_1, \dots, x_n; \xi_1, \dots, \xi_n)$, $(y; \eta) = (y_0, \dots, y_{n-1}; \eta_0, \dots, \eta_{n-1})$ of T^*U_X and T^*U_Y respectively, we also set

$$\Omega_X = \{(x_1, \dots, x_n; \xi_1, \dots, \xi_n) \in T^*U_X \mid \xi_n \neq 0\} \subset \dot{T}^*X,$$
$$\Omega_Y = \{(y_0, \dots, y_{n-1}; \eta_0, \dots, \eta_{n-1}) \in T^*U_Y \mid \eta_0 \neq 0\} \subset \dot{T}^*Y.$$

Then by the description of $S \cap (U_X \times U_Y)$ above, we can easily show that $\Phi(\Omega_X) = \Omega_Y$ and the isomorphism $\Phi|_{\Omega_X} : \Omega_X \xrightarrow{\sim} \Omega_Y$ is explicitly given by

$$\begin{cases} y_0 = -\displaystyle\sum_{i=1}^{n-1} \frac{x_i \xi_i}{\xi_n} - x_n, & \\ y_j = \dfrac{\xi_j}{\xi_n} & (j = 1, 2, \dots, n-1), \\ \eta_0 = -\xi_n, & \\ \eta_j = -\xi_n x_j & (j = 1, 2, \dots, n-1). \end{cases}$$

This is the classical Legendre transform.

Now by a projective linear transform of $X = \mathbb{P}_n$, we may assume that $\pi_X(\theta_x) = 0 \in M \cap U_X$ and $\theta_x = p_X := (0; (0, \dots, 0, 1)) = (0; +dx_n) \in \dot{T}^*_M U_X$. Set $m = \dim M < n$ and $x' = (x_1, \dots, x_m)$. Then we may assume further that M has the form

$$M \cap U_X = \{x \in U_X = \mathbb{R}^n_x \mid x_i = f_i(x') \ (i = m+1, \dots, n)\}$$

in an open neighborhood of $0 \in \mathbb{R}^n_x$, where f_i's are real analytic functions satisfying the conditions

$$f_i(0) = \partial_1 f_i(0) = \cdots = \partial_m f_i(0) = 0.$$

This means that the embedded tangent plane of M at $0 \in M$ (i.e. the linear subspace of $X = \mathbb{P}_n$ of dimension $m = \dim M$ having the same tangent space as M at $0 \in U_X$) is given by the equations

$x_{m+1} = \cdots = x_n = 0$ in U_X. By the assumption $\dim \mathcal{Z}_{\theta_x} = r$ for $\theta_x = p_X = (0; +dx_n)$, there exists a partition $p+q+r+s = n-1$ of $n-1$ satisfying $p+q+r = m = \dim M$ and positive real numbers $a_1, \ldots, a_{p+q}$ such that the function $f_n(x')$ is developed into a Taylor series as

$$f_n(x') = a_1 x_1^2 + \cdots + a_p x_p^2 - a_{p+1} x_{p+1}^2 - \cdots - a_{p+q} x_{p+q}^2 + (\text{higher order terms}).$$

In this situation, we have $\Phi(p_X) = (0; (-1, 0, \ldots, 0)) = (0; -dy_0) \in T^*U_Y$. We set $p_Y = \Phi(p_X)$.

Recall that the submanifold $N \subset Y$ satisfies the following property: There exists an open neighborhood W of p_X in $\dot{T}^*_M X$ such that $\Phi(W) \subset \dot{T}^*_N Y$. We describe this N in the affine open chart $U_Y = \mathbb{R}^n_{y_0,\ldots,y_{n-1}}$ and set $y'' = (y_1, \ldots, y_{p+q}, y_{m+1}, \ldots, y_{n-1})$. Then by an elementary differential calculus with the aid of the proof of [9, Proposition 2.5.3], the submanifold $N \subset Y$ has the form

$$N = \bigl\{ y \in U_Y \mid y_i = g_i(y'')\ (i = 0, p+q+1, \ldots, m) \bigr\}$$

in an open neighborhood of $0 \in U_Y$, where g_i's are real analytic functions satisfying

$$g_i(0) = 0 \quad \text{and} \quad \partial_j g_i(0) = 0 \quad \text{for } j = 1, \ldots, p+q, m+1, \ldots, n-1.$$

Moreover, there exist positive real numbers $b_1, \ldots, b_{p+q}$ such that the function $g_0(y'')$ is developed into a Taylor series as

$$g_0(y'') = b_1 y_1^2 + \cdots + b_p y_p^2 - b_{p+1} y_{p+1}^2 - \cdots - b_{p+q} y_{p+q}^2 + (\text{higher order terms}).$$

In particular, the dimension of N is $p+q+s = (n-1) - r$ as is stated in Proposition 4.1.

Now, let us consider the sheaf $\mathbb{C}_M \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ which corresponds to $\mathbf{1}_M \in CF(X)$. Instead of treating $\mathcal{R}_S(\mathbf{1}_M) \in CF(Y)$, we shall study the sheaf-theoretical Radon transform of $\mathbb{C}_M \in \mathbf{D}^b_{\mathbb{R}-c}(X)$

$$\mathcal{R}_S(\mathbb{C}_M) := Rg_* f^{-1}(\mathbb{C}_M) \in \mathbf{D}^b_{\mathbb{R}-c}(Y).$$

From now on, we make use of the theory of simple sheaves developed by Kashiwara–Schapira [19, Chapter VII]. Recall that $\mathbb{C}_M$ is a simple sheaf with shift $\frac{1}{2}\operatorname{codim} M = \frac{1}{2}(s+1)$ along $\dot{T}^*_M X$ at p_X. In order to calculate the shift of the sheaf-theoretical integral transform $\mathcal{R}_S(\mathbb{C}_M) \simeq Rp_{2!}(\mathbb{C}_S \otimes p_1^{-1}\mathbb{C}_M)$, by Proposition 7.5.6 of [19], we have to calculate the Maslov index (for the definition see [19, Appendix A.3]) of the following three Lagrangian planes in $T_{p_X}(T^*X)$.

$$\lambda_0(p_X) = T_{p_X}\bigl(\pi_X^{-1}\pi_X(p_X)\bigr),$$
$$\lambda_{T^*_M X}(p_X) = T_{p_X}(T^*_M X),$$
$$\lambda_1(p_X) = (\Phi'_{p_X})^{-1}\bigl(T_{p_Y}\bigl(\pi_Y^{-1}\pi_Y(p_Y)\bigr)\bigr).$$

Using the coordinate system $(x;\xi)$ of $T^*U_X \ni p_X$, let us identify $T_{p_X}(T^*X)$ with $\mathbb{R}^{2n}_{x,\xi}$. Also set $a_1 x_1^2 + \cdots + a_p x_p^2 - a_{p+1} x_{p+1}^2 - \cdots - a_{p+q} x_{p+q}^2 = \sum_{j=1}^m \frac{c_j}{2} x_j^2$. Then we have

$$\begin{aligned}
\lambda_0(p_X) &= \big\{(0,\dots,0;\xi_1,\dots,\xi_n) \mid \xi \in \mathbb{R}^n\big\},\\
\lambda_{T_M^*X}(p_X) &= \big\{(x_1,\dots,x_m,0,\dots,0;-c_1x_1,\dots,-c_mx_m,\xi_{m+1},\dots,\xi_n)\\
&\qquad \mid x' \in \mathbb{R}^m, \xi_{m+1},\dots,\xi_n \in \mathbb{R}\big\},\\
\lambda_1(p_X) &= \big\{(x_1,\dots,x_{n-1},0;0,\dots,0,\xi_n) \mid x_1,\dots,x_{n-1},\ \xi_n \in \mathbb{R}\big\}
\end{aligned}$$

and the Maslov index $\tau(\lambda_0(p_X), \lambda_{T_M^*X}(p_X), \lambda_1(p_X))$ is calculated as follows.

$$\tau\big(\lambda_0(p_X), \lambda_{T_M^*X}(p_X), \lambda_1(p_X)\big) = \sharp\{j \mid c_j < 0\} - \sharp\{j \mid c_j > 0\} = q - p.$$

Therefore by Proposition 7.5.6 of [19] $\mathcal{R}_S(\mathbb{C}_M) \in \mathbf{D}^b_{\mathbb{R}-c}(Y)$ is a simple sheaf with shift

$$\frac{1}{2}(s+1) - \frac{1}{2}(n-1) - \frac{1}{2}\tau\big(\lambda_0(p_X), \lambda_{T_M^*X}(p_X), \lambda_1(p_X)\big) = \frac{1}{2}(1-r-2q)$$

along T_N^*Y at p_Y. Since $\mathbb{C}_N \in \mathbf{D}^b_{\mathbb{R}-c}(Y)$ is a simple sheaf with shift $\frac{1}{2}\operatorname{codim} N = \frac{1}{2}(r+1)$, we obtain an isomorphism

$$\mathcal{R}_S(\mathbb{C}_M) \simeq \mathbb{C}_N[-q-r]$$

in the category $\mathbf{D}^b(Y; p_Y)$ (the localization of the derived category $\mathbf{D}^b(Y)$ at $p_Y \in \dot{T}^*Y$).

Hence we get

$$\mathcal{R}_S^{\mu}\big([T_M^*X]\big) = CC\big(\mathcal{R}_S(\mathbf{1}_M)\big) = CC\big(\mathcal{R}_S(\mathbb{C}_M)\big) = (-1)^{q+r}[T_N^*Y]$$

in an open neighborhood of p_Y. Together with the equality $I_M(\theta_x) = q + r$ for $\theta_x = p_X = (0; +dx_n) \in \dot{T}_M^*X$, this completes the proof of Theorem 4.3. □

Remark 4.4. Let $X = \mathbb{P}_n$, $Y = \mathbb{G}_{n,k}$ for $1 \leqslant k \leqslant n-1$, $S = \{(l, L) \in X \times Y \mid 0 \in l \subset L \subset \mathbb{R}^{n+1}\}$ and consider the topological Radon transform $\mathcal{R}_S : CF(X) \to CF(Y)$ defined in Section 3. Then also in this more general setting, we can prove some results analogous to Theorem 4.3. Indeed for a locally closed subanalytic submanifold $M \subset X$, the sheaf-theoretical Radon transform $\mathcal{R}_S(\mathbb{C}_M) = Rp_{2!}(\mathbb{C}_S \otimes p_1^{-1}\mathbb{C}_M) \in \mathbf{D}^b(Y)$ is a simple sheaf at generic points of $SS(\mathcal{R}_S(\mathbb{C}_M)) \cap \dot{T}^*Y$ thanks to [18, Corollary 7.3.5]. Hence the proof proceeds in the same way as in Theorem 4.3. We will develop this idea in the forthcoming paper [24]. The results that we obtain are related to associated varieties studied by Gelfand–Kapranov–Zelevinski [10].

The results obtained in this section can be summarized as follows.

Theorem 4.5. *Let $\varphi : X = \mathbb{P}_n \to \mathbb{Z}$ be a constructible function and $X = \bigsqcup_{\alpha \in A} X_\alpha$ a μ-stratification of X adapted to φ. For each stratum X_α satisfying* $\dim X_\alpha < n = \dim X$*, take an open dense subset Ω_α of $\dot{T}_{X_\alpha}^*X$ such that*

$$\Omega_\alpha \subset \dot{T}_{X_\alpha}^*X \Big\backslash \bigsqcup_{\beta \neq \alpha} \overline{T_{X_\beta}^*X}.$$

The map $\dot{\pi}_Y \circ \Phi|_{\Omega_\alpha} : \Omega_\alpha \to Y = \mathbb{P}_n^*$ *induces a submersion to an open subset of a locally closed subanalytic submanifold* $Y_\alpha \subset Y = \mathbb{P}_n^*$. *Then for any stratum* X_α *such that* $\dim X_\alpha < n = \dim X$, *we have*

(i) *The dimension of the space of nullity* $\mathcal{Z}_{\theta_x} \subset T_x X_\alpha$ *of the second fundamental form*

$$h_{X_\alpha,\theta_x}(\cdot,\cdot) : T_x X_\alpha \times T_x X_\alpha \to \mathbb{R} \tag{4.2}$$

(*with respect to the canonical metric of* $X = \mathbb{P}_n$) *does not depend on* $\theta_x \in \Omega_\alpha$ *and is equal to* $(n-1) - \dim Y_\alpha$.

(ii) *For each* $\theta_x \in \Omega_\alpha \subset \dot{T}^*_{X_\alpha} X$, *denote by* $I_{X_\alpha}(\theta_x)$ *the number of non-positive eigenvalues of the second fundamental form* (4.2) (*counted with multiplicities*). *Assume that*

$$CC(\varphi) = m_\alpha [T^*_{X_\alpha} X]$$

for an integer $m_\alpha \in \mathbb{Z}$ *in an open neighborhood of* θ_x. *Then we have*

$$CC(\mathcal{R}_S(\varphi)) = (-1)^{I_{X_\alpha}(\theta_x)} m_\alpha [T^*_{Y_\alpha} Y]$$

in an open neighborhood of $\Phi(\theta_x) \in \dot{T}^*_{Y_\alpha} Y$.

Remark 4.6. Although we devoted ourselves to studying the real case in Theorems 4.3 and 4.5, we can also easily recover the main results of Ernström [8] in the complex case along the same line. Indeed, the sign changes of characteristic cycles via the topological Radon transform between two complex projective spaces can be computed with the help of Exercise A.7 of Kashiwara–Schapira [19].

5. Applications to real projective duality

In this section, we apply the results obtained in previous sections to the duality of real projective varieties $M \subset \mathbb{P}_n$. When M is smooth, we show that the Euler–Poincaré indices of the hyperplane sections $M \cap H$ of M can be expressed in terms of the singularities of the dual $M^* \subset \mathbb{P}_n^*$ of M. More precisely, we obtain a result (see Theorems 5.13 and 5.14 below) which relates the tangency of a hyperplane H to M with the singularities of M^* at $H \in M^* \subset \mathbb{P}_n^*$. This is an analogue of Ernström [8, Corollary 3.9] in the real setting.

5.1. Some properties of Legendre singularities

Let X be a real analytic manifold. In this subsection, we prove some basic results on subanalytic subsets $Z \subset X$ defined as projections of closed conic subanalytic Lagrangian submanifolds $\Lambda \subset \dot{T}^* X$. Such singular sets Z are called Legendre singularities and studied by many people especially in relation with geometric optics.

Definition 5.1. We say that a conic subanalytic subset $\Lambda \subset \dot{T}^* X$ is projective if Λ is invariant by the antipodal map $\alpha_{T^*X} : \dot{T}^* X \xrightarrow{\sim} \dot{T}^* X$.

Note that $\Lambda \subset \dot{T}^* X$ is projective if and only if there exists a subanalytic subset Λ' in the projective cotangent bundle $P^* X = \dot{T}^* X / \mathbb{R}^\times$ such that $\Lambda = \gamma_X^{-1}(\Lambda')$ for $\gamma_X : \dot{T}^* X \twoheadrightarrow P^* X$.

Proposition 5.2. *Let $\Lambda \subset \dot{T}^*X$ be a closed conic subanalytic Lagrangian submanifold (in particular Λ is smooth). Assume that Λ is projective. Then*

(i) *$Z = \dot{\pi}_X(\Lambda) \subset X$ is a closed subanalytic subset such that $\Lambda = \overline{T^*_{Z_{\mathrm{reg}}}X} \cap \dot{T}^*X$, where Z_{reg} is the smooth part of Z.*
(ii) *Assume moreover that $\gamma_X(\Lambda) \subset P^*X$ is connected. Then Z is connected and equidimensional (i.e. the dimension $\dim_x Z$ of Z at $x \in Z_{\mathrm{reg}}$ does not depend on x).*

Proof. (i) Consider a closed subanalytic submanifold $\Lambda' = \gamma_X(\Lambda) \subset P^*X$ which satisfies $\Lambda = \gamma_X^{-1}(\Lambda')$. Then we have $Z = \pi'(\Lambda')$ for $\pi' : P^*X \to X$. Since π' is proper, Z is a closed subanalytic set.

Let us prove the remaining assertion when Λ' is connected. We take a μ-stratification (see [19, Chapter VIII]) $X = \bigsqcup_{\alpha\in A} X_\alpha$ of X refining the partition $X = (X \setminus Z) \sqcup Z_{\mathrm{reg}} \sqcup Z_{\mathrm{sing}}$ and satisfying the condition $\Lambda \subset \bigsqcup_{\alpha\in A} T^*_{X_\alpha}X$, where we set $Z_{\mathrm{sing}} = (Z \setminus Z_{\mathrm{reg}})$. Then we can choose a subset $\{\alpha_i\}_{i\in I} \subset A$ such that

(a) $X_{\alpha_i} \subset Z_{\mathrm{reg}}$ and $\dim X_{\alpha_i} = \dim_x Z_{\mathrm{reg}}$ for any $x \in X_{\alpha_i}$,
(b) $\bigcup_{i\in I} \overline{X_{\alpha_i}} = Z$,

because Z_{reg} is open dense in Z. For each X_{α_i} ($i \in I$), there exists an open neighborhood U of X_{α_i} in X such that

$$\Lambda \cap \dot{\pi}_X^{-1}(U) \subset \dot{\pi}_X^{-1}(X_{\alpha_i}).$$

Since Λ is a conic Lagrangian submanifold of $\dot{T}^*X$, it follows from Kashiwara's famous argument (see the proof of Proposition 8.3.10 of [19]) that

$$\Lambda \cap \dot{\pi}_X^{-1}(U) \subset \dot{T}^*_{X_{\alpha_i}}X.$$

Together with the assumption that Λ is projective and $\dot{\pi}_X(\Lambda \cap \dot{\pi}_X^{-1}(U)) = X_{\alpha_i}$ ($X_{\alpha_i} \subset Z = \dot{\pi}_X(\Lambda)$), we obtain

$$\Lambda \cap \dot{\pi}_X^{-1}(X_{\alpha_i}) = \dot{T}_{X_{\alpha_i}}X.$$

Therefore the inclusion

$$\overline{T^*_{Z_{\mathrm{reg}}}X} = \bigcup_{i\in I} \overline{T^*_{X_{\alpha_i}}X} \subset \Lambda$$

in $\dot{T}^*X$ was proved. In order to prove that this is in fact an equality, denote by d the maximal rank of the map $\dot{\pi}_X$ on Λ. Since $\Lambda' = \gamma_X(\Lambda)$ is assumed to be connected and the map $\dot{\pi}_X$ is real analytic, there exists an open dense subset Ω_Λ of Λ such that the rank of $\dot{\pi}_X$ is d at any point of Ω_Λ. This shows that all strata X_{α_i}'s for $i \in I$ are d-dimensional and

$$\overline{T^*_{Z_{\mathrm{reg}}}X} = \bigcup_{i\in I} \overline{T^*_{X_{\alpha_i}}X}$$

coincides with Λ in $\dot{T}^*X$. Note that in this case Z is equidimensional.

Finally, let us treat the case where $\Lambda' = \gamma_X(\Lambda)$ is not necessarily connected. First, we decompose $\Lambda' \subset P^*X$ into connected components as $\Lambda' = \bigsqcup_{j \in J} L_j$ and set $\Lambda_j = \gamma_X^{-1}(L_j)$ and $Z_j = \dot{\pi}_X(\Lambda_j)$. Then by the result above, we have

$$\overline{T^*_{Z_{j,\mathrm{reg}}} X} = \Lambda_j \quad \text{for each } \Lambda_j.$$

Hence it suffices to prove that for each $j \in J$ the set $Z_{j,\mathrm{reg}} \setminus (\bigcup_{j' \neq j} Z_{j'})$ is open dense in $Z_{j,\mathrm{reg}}$.

Lemma 5.3. *For $j, j' \in J$ satisfying $j \neq j'$, we have*

$$\dim(Z_{j,\mathrm{reg}} \cap Z_{j',\mathrm{sing}}) < \dim Z_{j,\mathrm{reg}}.$$

Proof. To prove this by a contradiction, assume first that $\dim(Z_{j,\mathrm{reg}} \cap Z_{j',\mathrm{sing}}) = \dim Z_{j,\mathrm{reg}}$. Take a μ-stratification $Z_{j'} = \bigsqcup_{\beta \in B} Z_{j',\beta}$ of $Z_{j'}$ such that there exists $\beta \in B$ satisfying the conditions $Z_{j',\beta} \subset Z_{j,\mathrm{reg}} \cap Z_{j',\mathrm{sing}}$ and $\dim Z_{j',\beta} = \dim Z_j$. Then by a fundamental property of μ-stratifications, $\Lambda_{j'} = \overline{T^*_{Z_{j',\mathrm{reg}}} X}$ must intersects with $\dot{T}^*_{Z_{j',\beta}} X$. Since we have

$$\dot{T}^*_{Z_{j',\beta}} X \subset \dot{T}^*_{Z_{j,\mathrm{reg}}} X \subset \Lambda_j,$$

this contradicts our assumption $\Lambda_j \cap \Lambda_{j'} = \emptyset$. □

Let us come back to the proof of Proposition 5.2. For $j \neq j'$, it follows also from $\Lambda_j \cap \Lambda_{j'} = \emptyset$ in Lemma 5.3 that $Z_{j',\mathrm{reg}}$ must intersect with $Z_{j,\mathrm{reg}}$ transversally. Together with the previous result $\dim(Z_{j,\mathrm{reg}} \cap Z_{j',\mathrm{sing}}) < \dim Z_{j,\mathrm{reg}}$, this implies that $Z_{j,\mathrm{reg}} \setminus Z_{j'}$ is open dense in $Z_{j,\mathrm{reg}}$. Since the family of sets $\{Z_{j'}\}_{j' \neq j}$ is locally finite by the real-analyticity of Λ, this completes the proof of (i).

(ii) Since Z is the image of the connected manifold $\Lambda' = \gamma_X(\Lambda) \subset P^*X$ by $\pi' : P^*X \to X$, Z is also connected. The equidimensionality of Z was already proved in (i). □

Remark 5.4. In the C^∞-category, a result very close to the proposition above was obtained also in Ishikawa–Morimoto [14].

5.2. Dual varieties in the real analytic case

Now let us return our original situation: $X = \mathbb{P}_n$ and $Y = \mathbb{P}_n^*$, etc. We consider a real analytic submanifold M of X. Then since the conormal bundle $\dot{T}^*_M X$ is projective in the sense of Definition 5.1, $(\dot{\pi}_Y \circ \Phi)(\dot{T}^*_M X)$ is a closed subanalytic subset of Y by Proposition 5.2. If moreover M is connected, the image of the Lagrangian submanifold $\Phi(\dot{T}^*_M X) \subset \dot{T}^*Y$ by the projection $\gamma_Y : \dot{T}^*Y \twoheadrightarrow P^*Y$ is connected, too. Therefore by Proposition 5.2, the subanalytic set $(\dot{\pi}_Y \circ \Phi)(\dot{T}^*_M X) \subset Y$ is equidimensional.

Definition 5.5. Let $M \subset X = \mathbb{P}_n$ be a connected real analytic submanifold. We call the closed subanalytic set $M^* = (\dot{\pi}_Y \circ \Phi)(\dot{T}^*_M X)$ the dual of M.

We shall explain a method to facilitate the computation of the dual M^* and give some examples.

Using the notations in Section 3, we first clarify the geometry of the diagram

$$\begin{tikzcd} & \dot{T}^*_S(X \times Y) \arrow[dl, "\widetilde{p_1}"', "\sim"] \arrow[dr, "\widetilde{p_2}^a", "\sim"'] & \\ \dot{T}^*X & & \dot{T}^*Y \end{tikzcd}$$

for the case of $X = \mathbb{P}_n$, $Y = \mathbb{G}_{n,n-1} \simeq \mathbb{P}_n^*$ and $S = \{(l, H) \in X \times Y \mid 0 \in l \subset H\}$. By projectivizing this diagram we obtain

$$\begin{tikzcd} & P^*_S(X \times Y) \arrow[dl, "\check{p}_1"', "\sim"] \arrow[dr, "\check{p}_2^a", "\sim"'] & \\ P^*X & & P^*Y. \end{tikzcd} \tag{5.1}$$

The projective cotangent bundle $P^*X = \dot{T}^*X/\mathbb{R}^\times$ in (5.1) can be naturally identified with the incidence submanifold $S \subset X \times Y$ in the following way. For the pair $(l, H) \in S$, consider the point $x_0 = \check{l} \in X = \mathbb{P}_n$ (respectively the hyperplane $\check{H} \subset X = \mathbb{P}_n$ passing through x_0) which corresponds to l (respectively H). Then we get a point $(T^*_{\check{H}}X)_{x_0} \in P^*X$. We thus obtained an isomorphism $\rho_X : S \xrightarrow{\sim} P^*X$. Since $Y = \mathbb{G}_{n,n-1}$ is identified with the dual projective space $\mathbb{P}_n^* = (\{\mathbb{R}^{n+1}\}^* \setminus \{0\})/\mathbb{R}^\times$, similarly we have an isomorphism $\rho_Y : S \xrightarrow{\sim} P^*Y$. The following lemma is well known.

Lemma 5.6. *The diagram*

$$\begin{tikzcd} P^*X & P^*_S(X \times Y) \arrow[l, "\check{p}_1"', "\sim"] \arrow[d, "\check{p}_2^a", "\wr"'] \\ S \arrow[u, "\rho_X", "\wr"'] \arrow[r, "\sim", "\rho_Y"'] & P^*Y \end{tikzcd}$$

is commutative.

The proof of this lemma immediately follows from the explicit description of the map $\Phi = (\widetilde{p_2}^a \circ \widetilde{p_1}^{-1}) : \dot{T}^*X \xrightarrow{\sim} \dot{T}^*Y$ in the proof of Theorem 4.3. Indeed, in standard affine charts of $X = \mathbb{P}_n$ and $Y = \mathbb{P}_n^*$, the map Φ is nothing but the classical Legendre transform. Furthermore in the homogeneous coordinates $[a] = [a_0 : a_1 : \cdots : a_n]$ and $[b] = [b_0 : b_1 : \cdots : b_n]$ of X and Y respectively, the projectivization $\check{\Phi} = (\check{p}_2^a \circ \check{p}_1^{-1}) : P^*X \xrightarrow{\sim} P^*Y$ of Φ is simply given by $([a]; [b]) \mapsto ([b]; [a])$ ($\sum_{j=0}^n a_j b_j = 0 \Leftrightarrow ([a]; [b]) \in S$). This shows the triviality of the map $\check{\Phi}$.

Now, let $M \subset X = \mathbb{P}_n$ be a connected real analytic submanifold as before. Then by the above transparent description of the map $\check{\Phi} : P^*X \xrightarrow{\sim} P^*Y$, in order to compute the dual M^* of M, it suffices to take the projectivized conormal bundle $P^*_M X$ in $P^*X \simeq P^*Y$ and push it down to $Y = \mathbb{P}_n^*$. The same argument in the complex case is well known and can be found, for example, in Fischer–Piontkowski [9].

Let us give also some examples of dual sets. As we see in the examples below, even if we start from a smooth submanifold $M \subset X$, the dual M^* may become very singular. In the complex case, we know moreover that the dual of a smooth projective variety is almost always singular.

Example 5.7.

(i) Let $\iota_n : \mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ be the real Veronese embedding given by $[x : y] \mapsto [x^n : x^{n-1}y : \cdots : xy^{n-1} : y^n]$ and set $M = \iota_n(\mathbb{P}_1) \subset \mathbb{P}_n$. Then the dual $M^* \subset \mathbb{P}_n^*$ is a hypersurface defined by the classical discriminant for polynomials of degree n.

(ii) For $n \geqslant m$, we consider the real Segre embedding $\iota_{n,m} : \mathbb{P}_n \times \mathbb{P}_m \hookrightarrow \mathbb{P}_{(n+1)(m+1)-1}$ given by $([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) \mapsto [\cdots : x_i y_j : \cdots]$. Set $M = \iota_{n,m}(\mathbb{P}_n \times \mathbb{P}_m) \subset \mathbb{P}_{(n+1)(m+1)-1}$. Then the dual defect $\delta_*(M)$ of M (see (5.2) below) is $n - m$. Indeed, let $M_{(n+1),(m+1)}$ be the space of real $(n+1) \times (m+1)$ matrices and identify the dual projective space $\mathbb{P}^*_{(n+1)(m+1)-1}$ with its projectivization $\mathbb{P}(M_{(n+1),(m+1)})$. Then the dual $M^* \subset \mathbb{P}^*_{(n+1)(m+1)-1}$ is explicitly written by

$$M^* = \mathbb{P}\big(\{A \in M_{(n+1),(m+1)} \mid \operatorname{rank} A \leqslant m\}\big).$$

Therefore the dual M^* has a stratification defined by the ranks of matrices. From this, we see that the dual M^* has very complicated singularities.

As we have already observed in Proposition 4.1, the dimension of the dual M^* is determined by the non-degeneracy of the curvature of M. Namely, using the space of nullity $\mathcal{Z}_{\theta_x} \subset T_x M$ of the second fundamental form

$$h_{M,\theta_x}(\cdot,\cdot) : T_x M \times T_x M \to \mathbb{R}$$

for each $\theta_x \in \dot{T}^*_M X$ (see Section 4), we have

$$\dim M^* = (n-1) - \min\{\dim \mathcal{Z}_{\theta_x} \mid \theta_x \in \dot{T}^*_M X\}.$$

This implies that the codimension of the dual M^* in $Y = \mathbb{P}_n^*$ is one in almost all cases. Hence we define the dual defect $\delta_*(M) \geqslant 0$ of M by

$$\delta_*(M) = (n-1) - \dim M^* = \min\{\dim \mathcal{Z}_{\theta_x} \mid \theta_x \in \dot{T}^*_M X\}. \tag{5.2}$$

Similarly for the dual $M^* \subset Y$, we consider the second fundamental form

$$h_{M^*_{\mathrm{reg}},\theta_y}(\cdot,\cdot) : T_y M^*_{\mathrm{reg}} \times T_y M^*_{\mathrm{reg}} \to \mathbb{R}$$

associated with $\theta_y \in \dot{T}^*_{M^*_{\mathrm{reg}}} Y$ and its space of nullity $\mathcal{Z}'_{\theta_y} \subset T_y M^*_{\mathrm{reg}}$. We take and fix an open dense projective subset $\Omega_{M^*} \subset \dot{T}^*_{M^*_{\mathrm{reg}}} Y$ such that

$$\dim \mathcal{Z}'_{\theta_y} = \min\{\dim \mathcal{Z}'_{\theta_{y'}} \mid \theta_{y'} \in \dot{T}^*_{M^*_{\mathrm{reg}}} Y\}$$

for any $\theta_y \in \Omega_{M^*}$.

Definition 5.8. For each point $\theta_y \in \Omega_{M^*}$, we denote by $j_{M^*}(\theta_y)$ the number (counted with multiplicities) of positive eigenvalues of the second fundamental form

$$h_{M^*_{\mathrm{reg}},\theta_y}(\cdot,\cdot): T_y M^*_{\mathrm{reg}} \times T_y M^*_{\mathrm{reg}} \to \mathbb{R}$$

and set $J_{M^*}(\theta_y) = j_{M^*}(\theta_y) + \delta_*(M)$.

From now on, for the connected real analytic submanifold $M \subset X = \mathbb{P}_n$, we study the topological Radon transform $\varphi := \mathcal{R}_S(\mathbf{1}_M) \in CF(Y)$ of the function $\mathbf{1}_M \in CF(X)$. This constructible function φ on $Y = \mathbb{P}^*_n$ is especially important because its value at $y = H \in Y = \mathbb{P}^*_n$ is the Euler characteristic $\chi(M \cap H)$ of the hyperplane section $M \cap H$ of M.

Theorem 5.9.

(i) *Outside the zero-section of T^*Y, the set $\overline{T^*_{M^*_{\mathrm{reg}}} Y}$ is a smooth Lagrangian submanifold of T^*Y and coincides with $|CC(\varphi)|$.*
(ii) *In an open neighborhood of $\theta_y \in \Omega_{M^*}$, we have the equality*

$$CC(\varphi) = (-1)^{J_{M^*}(\theta_y)}[T^*_{M^*_{\mathrm{reg}}} Y].$$

Proof. The assertion (i) is an immediate consequence of Proposition 5.2 and the definition of the dual M^* of M. Also the assertion (ii) easily follows from the proof of Theorem 4.3. Indeed, in the notations of Section 4.3 (see Definition 4.2), we have $J_{M^*}(\theta_y) = I_M(\theta_x)$ where $\theta_x = \Phi^{-1}(\theta_y) \in \dot{T}^*_M X$ is the corresponding microlocal point. $\square$

5.3. A refinement of Euler integrals

In order to state our main theorems, we prepare some auxiliary results on constructible functions in this subsection.

Let $\varphi : X \to \mathbb{Z}$ be a constructible function on a real analytic manifold X and $U \subset X$ an open subanalytic subset. If the support of φ is compact in U, then we know that the Euler integral $\int_U \varphi$ of φ over U is well defined. We can slightly extend this definition to some cases where the support of φ is not necessarily compact in U. Namely, assuming that $U \subset X$ is a relatively compact open subanalytic subset in X, we have the following definition.

Definition 5.10. Let $U \subset X$ be a relatively compact open subanalytic subset in X. For $\varphi \in CF(X)$, we set

$$\int_U \varphi := \int_X \mathbf{1}_U \cdot \mathsf{D}_X \varphi,$$

where $\mathsf{D}_X\varphi \in CF(X)$ is a constructible function defined by

$$\mathsf{D}_X\varphi(x) = \lim_{\varepsilon \to +0} \int_X \mathbf{1}_{B(x,\varepsilon)} \cdot \varphi$$

for each $x \in X$ ($B(x, \varepsilon)$ is an open ball with radius ε centered at x).

Recall that the above operation $\mathsf{D}_X : CF(X) \to CF(X)$ corresponds to the Verdier duality functor $D_X : \mathbf{D}^b_{\mathbb{R}-c}(X) \xrightarrow{\sim} \mathbf{D}^b_{\mathbb{R}-c}(X)$. The next proposition justifies the definition above.

Proposition 5.11. *Let U be a relatively compact open subanalytic subset in X and $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$. Consider the constructible function $\chi(F) \in CF(X)$ obtained from F by taking the local Euler–Poincaré indices* (*see Theorem* 2.5). *Then we have*

$$\int_U \chi(F) = \sum_{j\in\mathbb{Z}} (-1)^j \dim H^j(U; F) = \chi(U; F).$$

Proof. Set $j : U \hookrightarrow X$ and $G = R\Gamma_U(F) \simeq Rj_* j^{-1} F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$. Then we have

$$R\Gamma(U; F) \simeq R\Gamma\bigl(X; R\Gamma_U(F)\bigr) = R\Gamma(X; G).$$

Since $R\Gamma(X; G) \simeq R\Gamma_c(X; G)$ and $R\Gamma(X; D_X G)$ are dual to each other, we get also $\chi(X; G) = \chi(X; D_X G)$. Therefore applying $\chi : \mathbf{K}_{\mathbb{R}-c}(X) \xrightarrow{\sim} CF(X)$ to the isomorphisms

$$D_X G = D_X R\Gamma_U(F) \simeq D_X D_X(\mathbb{C}_U \otimes D_X F) \simeq \mathbb{C}_U \otimes D_X F,$$

we obtain $\chi(D_X G) = \mathbf{1}_U \cdot \mathsf{D}_X(\chi(F))$ and the result follows. □

5.4. Reconstruction theorems: Main theorems

Let us now introduce our main theorems which enable us to completely reconstruct the whole function $\varphi = \mathcal{R}_S(\mathbf{1}_M)$ from its value at a point $y = H \in Y \setminus M^*$ and the singularity of the dual set M^*. Namely, we shall describe the topological jump

$$\chi(M \cap H') - \chi(M \cap H) = \varphi(H') - \varphi(H)$$

at $y' = H' \in M^*$ in terms of the singularity of M^* at H'. Although we cannot define Euler obstructions (see [8,16,20], etc. for the definition) for subanalytic sets, we obtain the following results surprisingly similar to the ones in [8]. Let us take and fix a μ-stratification $Y = \bigsqcup_{\alpha\in A} Y_\alpha$ of Y refining the dual set $M^* \subset Y$ (we always assume that all strata in a μ-stratification are connected). Then we have

Proposition 5.12. *The function $\varphi = \mathcal{R}_S(\mathbf{1}_M)$ is constant on each stratum Y_α.*

Proof. Consider the $\mathbb{R}$-constructible sheaf $F = Rg_* f^{-1}\mathbb{C}_M \in \mathbf{D}^b_{\mathbb{R}-c}(Y)$ on Y satisfying $\chi(F) = \varphi = \mathcal{R}_S(\mathbf{1}_M)$. Since F is a simple sheaf in the sense of Chapter VII of [19] outside the zero-section of T^*Y, the micro-support $SS(F)$ of F satisfies the condition

$$SS(F) \cap \dot{T}^*Y = \bigl|CC(\varphi)\bigr| \cap \dot{T}^*Y = \overline{T^*_{M^*_{\mathrm{reg}}} Y} \cap \dot{T}^*Y.$$

From this, we obtain

$$SS(F) \subset \bigsqcup_{\alpha\in A} T^*_{Y_\alpha} Y$$

and the result follows from Proposition 8.4.1 of [19]. □

By this proposition, we denote by $\varphi_\alpha \in \mathbb{Z}$ the value of the function $\varphi = \mathcal{R}_S(\mathbf{1}_M)$ on the stratum Y_α. Thanks to the next theorem, we can recursively determine the values $\{\varphi_\alpha\}_{\alpha\in A}$ of the function φ by induction on the codimensions of Y_α's in Y.

Theorem 5.13. *Let* $k \geqslant \operatorname{codim} M^*$. *Suppose that we have already determined the values φ_α's of φ on all strata Y_α's satisfying* $\operatorname{codim} Y_\alpha \leqslant k$ *and let Y_β be a stratum such that* $\operatorname{codim} Y_\beta = k+1$. *For a point $y \in Y_\beta$, choose a real-valued real analytic function ψ defined on a neighborhood of y satisfying the conditions $\psi^{-1}(0) \supset Y_\beta$ and*

$$\big(y; \operatorname{grad}\psi(y)\big) \in \dot{T}^*_{Y_\beta}Y \setminus \bigcup_{\alpha\neq\beta} \overline{T^*_{Y_\alpha}Y}.$$

Then the value φ_β of φ on Y_β is given by

$$\varphi_\beta = \int_{B(y,\varepsilon)\cap\{\psi<0\}} \varphi \tag{5.3}$$

for sufficiently small $\varepsilon > 0$. Here we used the Euler integral defined in Definition 5.10.

Proof. As in the proof of Proposition 5.12, let us consider the $\mathbb{R}$-constructible sheaf $F = Rg_* f^{-1}\mathbb{C}_M \in \mathbf{D}^b_{\mathbb{R}-c}(Y)$ such that $\chi(F) = \varphi$. Then by our assumption $(y; \operatorname{grad}\psi(y)) \notin \bigcup_{\alpha\neq\beta} \overline{T^*_{Y_\alpha}Y}$ and Theorem 5.9 (i), we obtain

$$\big(y; \operatorname{grad}\psi(y)\big) \notin \big|CC(\varphi)\big| \cap \dot{T}^*Y = SS(F) \cap \dot{T}^*Y.$$

In particular, this implies that $\chi(R\Gamma_{\{\psi\geqslant 0\}}(F)_y) = 0$ (to obtain this result, we may also apply the simplest case of the main theorem of [28] to the $\mathbb{R}$-constructible sheaf F). Therefore for sufficiently small $\varepsilon > 0$, we obtain

$$\begin{aligned}\varphi_\beta &= \chi(F_y) = \chi\big(R\Gamma\big(B(y,\varepsilon)\cap\{\psi<0\}; F\big)\big) = \int_{B(y,\varepsilon)\cap\{\psi<0\}} \chi(F)\\ &= \int_{B(y,\varepsilon)\cap\{\psi<0\}} \varphi.\end{aligned}$$

This completes the proof. □

Note that when we denote $B(y,\varepsilon)\cap\{\psi<0\}$ by simply B, we can rewrite (5.3) as

$$\varphi_\beta = \sum_{\alpha:\ Y_\alpha\cap B\neq\emptyset} \varphi_\alpha\big\{\chi(\overline{Y_\alpha}\cap B) - \chi(\partial Y_\alpha \cap B)\big\}$$

by using the topological Euler characteristics. So our recursive formula above for the description of the function $\varphi = \mathcal{R}_S(\mathbf{1}_M)$ is very similar to the one used in the definition of Euler obstructions in the complex case. For example, compare the statement of Theorem 5.13 with the definition of Euler obstructions in Kashiwara [16].

Now assume that the value $\varphi_{\alpha_0} \in \mathbb{Z}$ of the function $\varphi = \mathcal{R}_S(\mathbf{1}_M)$ on a stratum Y_{α_0} such that $Y_{\alpha_0} \subset Y \setminus M^*$ is already given. Since the function φ is locally constant on $Y \setminus M^*$, this is equivalent to give the value of φ on a connected component of $Y \setminus M^*$. If the codimension of the dual M^* in Y is one, the values of φ on various connected components of $Y \setminus M^*$ may be different from each other. This shows that the real projective duality is much subtler than the complex case studied by Ernström [8]. We can, however, describe the differences in terms of the curvature of M^* completely, by the next theorem which is an immediate consequence of Theorem 5.9(ii).

Theorem 5.14.

(i) *Suppose that* $\dim M^* < n-1$, *i.e.* $\delta_*(M) > 0$. *Then on* $Y \setminus M^*$, *the function* $\varphi = \mathcal{R}_S(\mathbf{1}_M)$ *takes the constant value* φ_{α_0}. *Furthermore, for a stratum* Y_α *such that* $Y_\alpha \subset M^*_{\mathrm{reg}}$ *and* $\dim Y_\alpha = \dim M^*$, *the value* φ_α *on* Y_α *is equal to* $\varphi_{\alpha_0} + (-1)^{J_{M^*}(\theta_y)}$, *where* θ_y *is a point in* $\dot{T}^*_{Y_\alpha} Y \cap \Omega_{M^*}$.

(ii) *Suppose that* M^* *is a hypersurface in* Y. *For each point* $y \in Y \setminus M^*$, *take a smooth curve* $c_y : I = [0,1] \to Y$ *connecting a point in* Y_{α_0} *and* y *passing only* $Y \setminus M^*$ *and* $(n-1)$*-dimensional strata in* M^*_{reg}. *We assume that the curve* c_y *intersects transversally with* $(n-1)$*-dimensional strata in* M^*_{reg} *only at points in* $\pi_Y(\Omega_{M^*})$. *We start with the value* φ_{α_0} *at* $c_y(0) \in Y_{\alpha_0}$, *and whenever we pass through an* $(n-1)$*-dimensional stratum* $Y_\alpha \subset M^*_{\mathrm{reg}}$ *at a point* $y_\alpha \in Y_\alpha$ *we add the integer* $(-1)^{J_{M^*}(\theta_{y_\alpha})} - (-1)^{J_{M^*}(-\theta_{y_\alpha})}$, *where* $\theta_{y_\alpha} \in \Omega_{M^*} \subset \dot{T}^*_{M^*_{\mathrm{reg}}} Y$ *is a point lying over* y_α *such that* $\langle \theta_{y_\alpha}, c'_y(t) \rangle > 0$ $(c_y(t) = y_\alpha)$. *Then the value at* $c_y(1) = y$ *which we finally obtain is equal to* $\varphi(y)$.

(iii) *In the situation of* (ii), *also the values of* φ *on* $(n-1)$*-dimensional strata in* M^*_{reg} *can be determined by the following rule. Take such a stratum* $Y_\alpha \subset M^*_{\mathrm{reg}}$ *and consider the two strata* Y_{α_1} *and* Y_{α_2} *in* $Y \setminus M^*$ *such that* $Y_\alpha \subset \overline{Y_{\alpha_i}}$ $(i = 1, 2)$. *If* φ_{α_1} *and* φ_{α_2} *are different by* ± 2, *then* φ_α *is equal to* $\frac{1}{2}(\varphi_{\alpha_1} + \varphi_{\alpha_2})$. *Otherwise,* φ_{α_1} *and* φ_{α_2} *are the same, and we have* $\varphi_\alpha = \varphi_{\alpha_1} + (-1)^{J_{M^*}(\theta_y)}$ *for a point* θ_y *in* $\dot{T}^*_{Y_\alpha} Y \cap \Omega_{M^*}$.

5.5. Helgason's support theorem for constructible functions

To end this paper, we shall give an analogue of Helgason's support theorem (see Helgason's book [13]) for topological Radon transforms.

Theorem 5.15. *Let* $K \subset \mathbb{R}^n$ *be a compact convex subset and* $\varphi \in CF(\mathbb{R}^n)$ *a constructible function on* $\mathbb{R}^n$ *such that* $\{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}$ *is a compact set in* $\mathbb{R}^n$. *Assume that for any hyperplane* $H \subset \mathbb{R}^n$ *such that* $H \cap K = \emptyset$, *we have* $\int_H \varphi = 0$. *Then* $\mathrm{supp}(\varphi) = \{x \in \mathbb{R}^n \mid \varphi(x) \neq 0\}$ *is contained in* K.

Proof. We essentially follow the microlocal proof of Helgason's support theorem given by Boman–Quinto [3].

Set $X = \mathbb{P}_n$, $Y = \mathbb{P}^*_n$ and consider the standard affine open subset $U_X \simeq \mathbb{R}^n$ of X as before. Identifying $\mathbb{R}^n$ with U_X, we extend $\varphi \in CF(\mathbb{R}^n)$ to a constructible function $\widetilde{\varphi}$ on $X = \mathbb{P}_n$ so that $\widetilde{\varphi}|_{X \setminus U_X}$ is identically zero. Consider the topological Radon transform $\mathcal{R}_S(\widetilde{\varphi})$ of $\widetilde{\varphi}$. Then, by our assumption, Corollary 3.7 and the explicit description of the map $\Phi : \dot{T}^*X \xrightarrow{\sim} \dot{T}^*Y$ given in the proof of Theorem 4.3, for any hyperplane H in $\mathbb{R}^n$ such that $H \cap K = \emptyset$, the characteristic cycle $CC(\varphi)$ of φ satisfies the condition

$$\left|CC(\varphi)\right| \cap \dot{T}^*_H\mathbb{R}^n = \emptyset.$$

Using this result, we can easily prove our theorem as follows. For $d > 0$, define an open neighborhood of K by $O(K; d) = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, K) < d\}$. Assuming that $\operatorname{supp}(\varphi)$ is not contained in K, consider the smallest positive number $d > 0$ such that $\overline{O(K; d)} \supset \operatorname{supp}(\varphi)$. Then at any point $x \in \overline{O(K; d)} \setminus O(K; d)$, we can take a function $\psi : \mathbb{R}^n \to \mathbb{R}$ of the form $\psi(x) = a_0 + \sum_{i=1}^n a_i x_i$ ($a_i \in \mathbb{R}$) such that $\{x \in \mathbb{R}^n \mid \psi(x) \leqslant 0\} \supset \overline{O(K; d)} \supset \operatorname{supp}(\varphi)$ and the hyperplane $H = \{x \in \mathbb{R}^n \mid \psi(x) = 0\}$ defined by ψ satisfies $x \in H$ and $H \cap K = \emptyset$. Now by the simplest case of the main theorem of Schürmann [28], from $|CC(\varphi)| \cap \dot{T}^*_H\mathbb{R}^n = \emptyset$ we obtain $\chi(R\Gamma_{\{\psi \leqslant 0\}}(F)_x) = 0$ for $F \in \mathbf{D}^b_{\mathbb{R}-c}(\mathbb{R}^n)$ such that $\chi(F) = \varphi$. Therefore we get

$$\varphi(x) = \chi(F_x) = \chi\bigl(R\Gamma\bigl(B(x, \varepsilon) \cap \{\psi > 0\}; F\bigr)\bigr)$$

for sufficiently small $\varepsilon > 0$. Since $\varphi|_{\{\psi > 0\}}$ is identically zero, using Proposition 5.11, we finally obtain

$$\varphi(x) = \chi\bigl(R\Gamma\bigl(B(x, \varepsilon) \cap \{\psi > 0\}; F\bigr)\bigr) = 0.$$

This means that φ is zero in an open neighborhood of x. Applying the same arguments to each $x \in \overline{O(K; d)} \setminus O(K; d)$ we find that there exists $d' > 0$ such that $d' < d$ and $\overline{O(K; d')} \supset \operatorname{supp}(\varphi)$. This contradicts the minimality of $d > 0$. □

Remark 5.16. Theorem 5.15 does not hold for any constructible function on $\mathbb{R}^n$. Even for constructible functions having closed supports, there are some counterexamples. Let us give a very simple example. In $\mathbb{R}^2$ consider the following disjoint subsets.

$$K = \bigl\{(x_1, x_2) \in \mathbb{R}^2 \bigm| x_1^2 + x_2^2 \leqslant 2\bigr\},$$
$$A_1 = \bigl\{(x_1, x_2) \in \mathbb{R}^2 \bigm| x_1^2 + x_2^2 > 2,\ x_1 = 1\bigr\},$$
$$A_2 = \bigl\{(x_1, x_2) \in \mathbb{R}^2 \bigm| x_1^2 + x_2^2 > 2,\ x_1 = -1\bigr\}.$$

We define a constructible function $\varphi : \mathbb{R}^2 \to \mathbb{Z}$ on $\mathbb{R}^2$ by

$$\varphi(x) = \begin{cases} 1, & x \in K \cup A_1, \\ -1, & x \in A_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\operatorname{supp}(\varphi) = \{x \in \mathbb{R}^2 \mid \varphi(x) \neq 0\}$ is a non-compact closed subset of $\mathbb{R}^2$ and for any line l in $\mathbb{R}^2$ such that $l \cap K = \emptyset$ we have $\int_l \varphi = 0$. However $\operatorname{supp}(\varphi)$ is not contained in K in this case.

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